Algebraic meaning of transpose..

$A \in \mathbb{R}^{m \times n}$

rows
\# of equations

columns
\# of variables

transpose swaps rows with columns.. what does that mean?

Matrix-vector multiplication.

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \in \mathbb{R}^{m \times n}$$

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$x \in \mathbb{R}^n$

$$f(x) = Ax = \underbrace{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n}_{\text{column representation of matrix vector mult.}}$$

$$A^T = \begin{bmatrix} a_1^T \\ \hline a_2^T \\ \hline \vdots \\ \hline a_n^T \end{bmatrix} \in \mathbb{R}^{n \times m}$$

$y \in \mathbb{R}^m$

$$A^T y = \begin{bmatrix} a_1 \cdot y \\ a_2 \cdot y \\ \vdots \\ a_n \cdot y \end{bmatrix}$$

row representation of matrix mult.

Consider

distribute y

$$Ax \cdot y = (a_1 x_1 + a_2 x_2 + \cdots + a_n x_n) \cdot y = x_1 (a_1 \cdot y) + x_2 (a_2 \cdot y) + \cdots + x_n (a_n \cdot y)$$

$$= x \cdot \begin{bmatrix} a_1 \cdot y \\ a_2 \cdot y \\ \vdots \\ a_n \cdot y \end{bmatrix} = x \cdot A^T y$$

Conclusion

$$Ax \cdot y = x \cdot A^T y \quad \text{for all } x \in \mathbb{R}^n \text{ and } y \in \mathbb{R}^m$$

moving A to the other side of the dot product gives the transpose... this is sometimes called duality.

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{2\times 3} \qquad x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \in \mathbb{R}^3 \qquad y = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \in \mathbb{R}^2$$

$$Ax = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-2+3 \\ 4-5+6 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$A^T y = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2-4 \\ 4-5 \\ 6-6 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$$

$$Ax \cdot y = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 4-5 = -1$$

$$x \cdot A^T y = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} = -2+1 = -1$$

magically the same!

Another way to see why $Ax \cdot y = x \cdot A^T y$ is to write every thing in terms of their components and track where they go.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \in \mathbb{R}^m$$

Maybe enough to use $m=2$ and $n=3$.

Product of Transposes

$f: \mathbb{R}^n \to \mathbb{R}^m$ $\qquad$ $g: \mathbb{R}^p \to \mathbb{R}^n$

$f(x) = \underset{m\times n}{A}x$ $\qquad$ $g(x) = \underset{n\times p}{B}x$

$$(f\circ g)(x) = f(g(x)) = f(Bx) = A(Bx) = (\underbrace{AB}_{C})x \qquad C \in \mathbb{R}^{m\times p}$$

$\underset{m\times p}{C} = \underset{m\times n}{A}\,\underset{n\times p}{B}$ is the matrix-matrix product.

What is $(AB)^T$? or $C^T \in \mathbb{R}^{p\times m}$?

$$\underset{m\times p}{C}\underset{p}{x} \cdot y = x \cdot \underset{p\times m}{C^T}\underset{m}{y} \quad \text{for all } x \in \mathbb{R}^p \text{ and } y \in \mathbb{R}^m.$$

Therefore

$$x \cdot C^T y = Cx \cdot y = A(Bx)\cdot y = Bx \cdot A^T y = x \cdot B^T(A^T y)$$

$$x \cdot (AB)^T y = x \cdot (B^T A^T)\, y \quad \text{for all } x \in \mathbb{R}^p \text{ and } y \in \mathbb{R}^m.$$

Claim $(AB)^T = B^T A^T$ ....

$x = x_1 e_1 + x_2 e_2 + \cdots + x_p e_p$ where $e_i \in \mathbb{R}^p$ are the standard basis vectors...

In fact

$$\underbrace{e_i \cdot (AB)^T y}_{i\text{th component of } (AB)^T y} = \underbrace{e_i \cdot (B^T A^T)\, y}_{i\text{th component of } B^T A^T y} \quad \text{for } i = 1, \ldots, p$$

Since the components of $(AB)^T y$ are the same as $B^T A^T y$

thus $(AB)^T y = B^T A^T y$ for all $y \in \mathbb{R}^m$.

Since the correspondence between matrices and linear functions is one to one. Thus $(AB)^T$ and $B^T A^T$ represent the same functions means they are the same matrices.

$$\boxed{(AB)^T = B^T A^T}$$