Matrix factorizations

Define matrix multiplication as the matrix corresponding to the composition of functions.

$A \in \mathbb{R}^{n \times n}$ $\qquad$ $B \in \mathbb{R}^{n \times n}$

$f: \mathbb{R}^n \to \mathbb{R}^n$ $\qquad$ $g: \mathbb{R}^n \to \mathbb{R}^n$

$f(x) = Ax$ $\qquad$ $g(x) = Bx$

We know $f \circ g$ is a linear function since f and g were linear, so there is a matrix $C \in \mathbb{R}^{n \times n}$ such that

$$(f \circ g)(x) = Cx$$

we define matrix matrix multiplication so $C = AB$.

what is C?

$$(f \circ g)(x) = f(g(x)) = f(Bx) = A(Bx) = Cx = (AB)x$$

Note finding C is complicated. If you just want to plug one vector in to $(f \circ g)(x)$ its easier to compute

$$w = Bx \quad \text{and then} \quad (f \circ g)(x) = Aw$$

using two matrix-vector multiplications.

How difficult is it to find C?

$$C = \left[(f \circ g)(e_1) \;\vdots\; (f \circ g)(e_2) \;\vdots\; \cdots \;\vdots\; (f \circ g)(e_n) \right]$$

$$(f \circ g)(e_i) = f(g(e_i)) = A(Be_i)$$

using a matrix-vector multiplication n times...

Suppose we wanted to compute $(f \circ g)(x)$ for m different x's

find C and then compute Cx for m different x's.

$n \quad + \quad m$ matrix vector mult.

Simply compute $\underbrace{A(Bx)}_{2m}$ for m different x's — $2m$ matrix vector mult

When is

$$n + m < 2m$$

$$n < m$$

So if there are more than n inputs x's then it's better to precompute C. Otherwise not.

If you can multiply matrices, you can then think about factoring (e...

$$[A \mid e_1 \vdots e_2] = \left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

$\underbrace{r_2 \leftarrow r_2 - 3r_1}_{\text{mult by a matrix}}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [r_2 \leftarrow r_2 - 3r_1] \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = E_1$

echelon

$$\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

$\underbrace{r_1 \leftarrow r_1 + r_2}_{\text{multiplication by}}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $[r_1 \leftarrow r_1 + r_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = E_2$

$$\left[\begin{array}{cc:cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

$\underbrace{r_2 \leftarrow -\frac{1}{2} r_2}_{\text{multiplication by}}$

$[r_2 \leftarrow -\frac{1}{2} r_2] = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} = E_3$

$$\left[\begin{array}{cc:cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

Therefore. $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

mult these matrices...

$$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

$$\underbrace{[r_2 \leftarrow r_2 - 3r_1]}\,[A \mid I] = [U \mid ?]$$

more here.

from before

$$\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

work for matrix matrix mult...

$$\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}^{-1} = [r_2 \leftarrow r_2 - 3r_1]^{-1} = [r_2 \leftarrow r_2 + 3r_1] = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

Check the inverse

$$AA^{-1} = I \qquad \text{for } A \in \mathbb{R}^{n \times n}$$
$$A^{-1}A = I$$

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This is the identity..

inverse over here.

$$\cancel{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}\cancel{\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}}\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

Therefore, I've factored the augmented matrix as...

$$\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

Also

$$\underbrace{\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}}_{A} = \underbrace{\begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}1 & 2\\ 0 & -2\end{bmatrix}}_{U}$$

These matrices are called triangular... and the idea is triangular matrices are simpler than the original.

Note could factor A more using the other row operations...

Transposes and inverses.

Let $A \in \mathbb{R}^{n\times n}$ and pivot in every row (so invertible).

$u = Ax$ so $x = A^{-1}u$

$$x \cdot y = A^{-1}u \cdot y = u \cdot (A^{-1})^T y = Ax \cdot (A^{-1})^T y = x \cdot A^T (A^{-1})^T y$$

Therefore

$$x \cdot Iy = x \cdot A^T (A^{-1})^T y \qquad \text{for all } x, y \in \mathbb{R}^n$$

Hence $\quad A^T (A^{-1})^T = I$

which means $(A^T)^{-1} = (A^{-1})^T$.