Matrix Factorization:

1. $\boxed{A = L\,U}$

upper triangular (echelon form of A)

lower triangular (zeros in upper right corner)

algebraic properties

2. $A = Q\,R$

upper triangular

orthogonal matrix (preserves angles)

geometric

3. $A = S D S^{-1}$

diagonal matrix (all off diagonal entries 0)

invertible matrix (columns are eigenvectors of A).

4. $A = U \Sigma V$ ← orthogonal (preserves angles)

diagonal (singular values of A)

orthogonal (preserves angles)

Example $A = L\,U$ perform elimination steps to find the echelon form

$$A = \begin{array}{c} \begin{array}{ccc} p & & \end{array} \\ \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} \end{array}$$

$r_2 \leftarrow r_2 + r_1$

$r_3 \leftarrow r_3 - 2r_1$

$$\begin{array}{c} \begin{array}{ccc} & p & \end{array} \\ \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{bmatrix} \end{array}$$

$r_3 \leftarrow r_3 + 5r_2$

$$U = \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$[r_3 \leftarrow r_3 + 5r_2][r_3 \leftarrow r_3 - 2r_1][r_2 \leftarrow r_2 + r_1]\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$[r_3 \leftarrow r_3 - 2r_1][r_2 \leftarrow r_2 + r_1]\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} = [r_3 \leftarrow r_3 - 5r_2]\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$[r_2 \leftarrow r_2 + r_1]\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} = [r_3 \leftarrow r_3 + 2r_1][r_3 \leftarrow r_3 - 5r_2]\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}}_{A} = \underbrace{[r_2 \leftarrow r_2 - r_1][r_3 \leftarrow r_3 + 2r_1][r_3 \leftarrow r_3 - 5r_2]}_{\text{multiply these together to find } L}\underbrace{\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}}_{U}$$

row col row col row col

↓ magic

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} = L$$

Therefore

$$\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$[r_2 \leftarrow r_2 - r_1]\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}$$

$$[r_3 \leftarrow r_3 + 2r_1]\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}$$

$$[r_3 \leftarrow r_3 - 5r_2]\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&-5&0\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-5&0\end{bmatrix}}_{\text{only change } r_3} = \begin{bmatrix}1&0&0\\-1&1&0\\2&-5&1\end{bmatrix} = L$$

$$[r_2 \leftarrow \underbrace{r_2 - r_1}_{\text{doesn't involve } r_3}][r_3 \leftarrow r_3 + 2r_1][r_3 \leftarrow r_3 - 5r_2]$$

$\leftarrow$ why one can just plop the entries in place

Consider the original elimination steps

$$[r_3 \leftarrow r_3 + 5r_2][r_3 \leftarrow r_3 - 2r_1][r_2 \leftarrow r_2 + r_1]\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

dependency relation

$$= [r_3 \leftarrow r_3 + 5r_2][r_3 \leftarrow r_3 - 2r_1]\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}$$

$$= [r_3 \leftarrow r_3 + 5r_2]\begin{bmatrix}1&0&0\\1&1&0\\-2&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\1&1&0\\3&5&1\end{bmatrix}$$

CAN'T just plop the entries in place

$A = LU$

$$\overbrace{\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}}^{A} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}}^{L} \overbrace{\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}}^{U}$$

Why is this useful?

Solve $Ax = b$

$$L\underbrace{Ux}_{y} = b$$

first solve $Ly = b$ for y

second solve $Ux = y$ for x

to find solution to $Ax = b$.

System of systems of linear equations

$$\begin{cases} Ly = b \\ Ux = y \end{cases}$$

Try $b = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$

$$\begin{array}{ccc} y_1 & y_2 & y_3 \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$$

✓ Algebraic notation.

$$\begin{aligned} y_1 &= -7 \\ -y_1 + y_2 &= 5 \\ 2y_1 - 5y_2 + y_3 &= 2 \end{aligned}$$

$$\begin{aligned} y_1 &= -7 \\ y_2 &= 5 + y_1 = 5 - 7 = -2 \\ y_3 &= 2 - 2y_1 + 5y_2 \\ &= 2 + 14 - 10 = 6 \end{aligned}$$

Therefore $y = \begin{bmatrix} -7 \\ -2 \\ 6 \end{bmatrix}$

Now solve $Ux = y$

$$\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 6 \end{bmatrix}$$ for x

echelon form so can solve using substitution.