Let $A \in \mathbb{R}^{m \times n}$ $\quad f(x) = Ax$, $\; f: \mathbb{R}^n \to \mathbb{R}^m$

soln. to homogeneous problem

$$\operatorname{Nul} A = \{ x \in \mathbb{R}^n : Ax = 0 \} \subseteq \mathbb{R}^n$$

$$\operatorname{Col} A = \{ Ax : x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$$

range of the function $f(x) = Ax$

The Basis Theorem (That dimension is well defined)

If $\mathcal{B} = \{u_1, u_2, \dots, u_p\}$ is a basis for $H \subseteq \mathbb{R}^n$ and

$\mathcal{C} = \{v_1, v_2, \dots, v_q\}$ is another basis for H.

Then $p = q$ and $\dim H = p$ the common value of the number of vectors in any basis for H.

In summary $\mathcal{B} = \{u_1, u_2, \dots, u_p\}$ is a basis of H if

$H = \operatorname{Col} A$ and $\operatorname{Nul} A = \{0\}$ where $A = [u_1 \vdots u_2 \vdots \cdots \vdots u_p]$.

A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

Goal. Find a basis for Col A.

$$A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} \sim \begin{array}{c} \begin{array}{cccc} P & P & F & F \end{array} \\ \begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

row equivalent... i.e. do some row operations to get the echelon form...

$$\begin{array}{cccc} P & P & F & F \\ \end{array}$$
$$\begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Claim can write the columns for the free variables as a combination of the pivot columns.

$$\begin{bmatrix} 6 \\ 5 \\ 0 \end{bmatrix} = ? \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Match the pivots with the corresponding elements in the free columns

$$\begin{bmatrix} 6 \\ 5 \\ 0 \end{bmatrix} - 5 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix} = (-4) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus,

$$\underbrace{\begin{bmatrix} 6 \\ 5 \\ 0 \end{bmatrix}}_{\text{3rd column}} = -4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{1st column}} + 5 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{\text{2nd columns}}$$

row operation

- $r_i \leftarrow r_i - \alpha r_j \quad i \neq j$
- $r_i \leftrightarrow r_j \quad i \neq j$
- $r_i \leftarrow \alpha r_i \quad \alpha \neq 0$

Note: row operations don't change the relationships between the columns of a matrix. (because they do the same thing to each column).

There the relationship found between pivot columns and free columns also holds for the original matrix.

$$\begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 9 \\ 1 \\ 8 \end{bmatrix} = -4 \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix} + 5 \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$$

← dependency relation between the columns of A

Do the same thing for the 4th column (also a free variable)

$$\begin{array}{ccccc} \begin{bmatrix}-5\\-6\\0\end{bmatrix} & = & 7\begin{bmatrix}1\\0\\0\end{bmatrix} & + & (-6)\begin{bmatrix}2\\1\\0\end{bmatrix} \\ \text{4th col} & & \text{1st col} & & \text{2nd col.} \end{array}$$

I can use this to find a basis for the column space. How?

A basis need independent vectors

$$\begin{bmatrix}-2\\12\\-3\end{bmatrix} = 7\begin{bmatrix}4\\6\\3\end{bmatrix} - 6\begin{bmatrix}5\\5\\4\end{bmatrix}$$

← dependency relation between the columns of A

Since you can't write a pivot column in the echelon form in terms of other pivot columns (because the pivots are in different rows). Then the vectors in the original matrix corresponding to the pivot columns are independent.

$$\text{Col}\begin{array}{c} \begin{array}{cccc} P & P & F & F \end{array} \\ \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} \end{array} = \text{Span}\underbrace{\left\{ \begin{bmatrix}4\\6\\3\end{bmatrix}, \begin{bmatrix}5\\5\\4\end{bmatrix} \right\}}_{\text{Basis for Col } A.}$$

Q: Find a basis for Col A:

$$A = \begin{array}{c} \begin{array}{cccc} P & P & F & F \end{array} \\ \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} \end{array} \sim \begin{array}{c} \begin{array}{cccc} P & P & F & F \end{array} \\ \begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Basis is $\left\{ \begin{bmatrix}4\\6\\3\end{bmatrix}, \begin{bmatrix}5\\5\\4\end{bmatrix} \right\}$.

Have to chose vectors from A,

$$\text{Col}\begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{span}\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix} \right\}$$

Warning: The column space of A and the column space of the echelon form of A are different!