Therefore.

$$A^{-1} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} \det A_1(e_1) & \det A_1(e_2) & \cdots & \det A_1(e_n) \\ \det A_2(e_1) & \det A_2(e_2) & & \\ \vdots & & \ddots & \\ \det A_n(e_1) & \cdots & & \det A_n(e_n) \end{bmatrix}$$

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

entries are $\det A_i(e_j)$

confusing because i is usually a row

$$A_i(e_j) = \begin{array}{c} \\ \end{array}\begin{array}{ccccccc} & & & i\text{th column} & & & \\ \end{array}$$

$$A_i(e_j) = \begin{bmatrix} a_1 & \cdots & a_{i-1} & e_j & a_{i+1} & \cdots & a_n \end{bmatrix}$$

expand along the ith column.

cross out the pth row and ith column of the matrix.

$$\det A_i(e_j) = \sum_{p=1}^{n} (-1)^{p+i} e_{j,p} \det A_{pi}$$

recall

expand on ith rows

$$\det A = \underbrace{\sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}}_{i \text{ const} \ldots}$$

expand on jth column

$$\det A = \underbrace{\sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}}_{j \text{ const}}$$

← the pth entry of the vector e_j.

$$e_{j,p} = \begin{cases} 0 & \text{if } p \neq j \\ 1 & \text{if } p = j \end{cases}$$

So $p=j$ is the only term that is non-zero in the sum.

$$\det A_i(e_j) = (-1)^{j+i} \det A_{ji} = C_{ji}$$

from lecture 17

$\sum a_{ij} C_{ij}$ here i is constant

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

here j constant

$$\det A_i(e_j) = (-1)^{j+i} \det A_{ji} = C_{ji}$$

Therefore.

$$A^{-1} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ \vdots & \vdots & & \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} \det A_1(e_1) & \det A_1(e_2) & \cdots & \det A_1(e_n) \\ \det A_2(e_1) & \det A_2(e_2) & & \\ \vdots & & \ddots & \\ \det A_n(e_1) & \cdots & & \det A_n(e_n) \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & & \vdots \\ \vdots & & \ddots & \\ C_{1n} & \cdots & & C_{nn} \end{bmatrix}$$

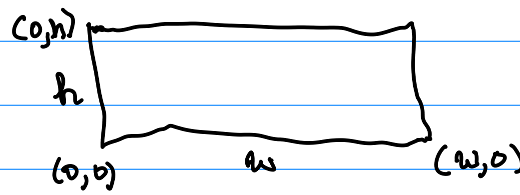
col row

order of the indices are reversed

Connection between determinants and volume elements...

Simplicity let $n=2$

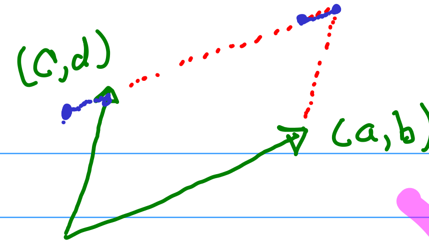
Area of a rectangle



$$A = \begin{bmatrix} w & 0 \\ 0 & h \end{bmatrix} \qquad \det A = wh = \text{area of the rectangle.}$$

Idea this works for any 2x2 matrix. I get the area of the parallelogram. Why?

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

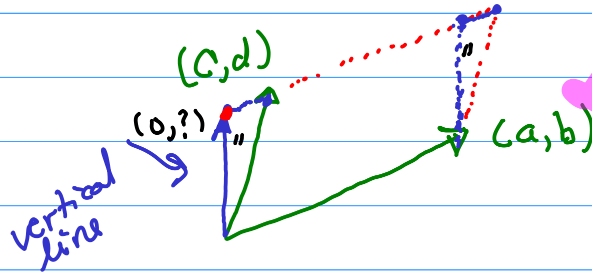


$r_2 \leftarrow r_2 - \frac{c}{a} r_1$

det of these matrices don't change because they are given by elimination operations.

these areas don't change

$$\begin{bmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{bmatrix}$$

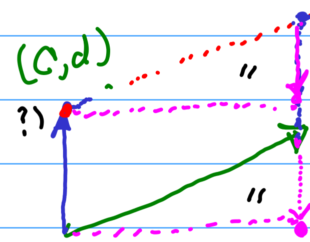


Want to subtract

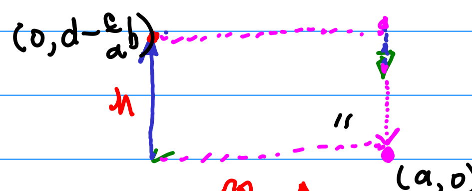
$(c,d) - \frac{c}{a}(a,b) = (0, d - \frac{c}{a} b)$

So I get a vertical line.

$r_1 \leftarrow r_1 - \frac{b}{d - \frac{c}{a} b} r_2$



$$\begin{bmatrix} a & 0 \\ 0 & d - \frac{c}{a} b \end{bmatrix}$$

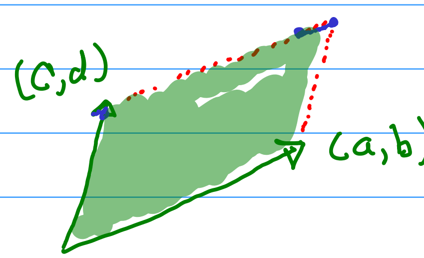


$$\det \begin{bmatrix} a & 0 \\ 0 & d - \frac{c}{a} b \end{bmatrix} = hw = \text{Area of this rectangle}$$

obvious...

So then

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \text{area of}$$



This works in any dimension...

Chapter 4 ... more about subspaces... $A \in \mathbb{R}^{m \times n}$

$$\text{Col } A = \{Ax : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

$$\text{Nul } A = \{x \in \mathbb{R}^n : Ax = 0\} \subseteq \mathbb{R}^n$$

$$\text{row } A = \text{Col } A^T = \{A^T z : z \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$$

Example:

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$$