$\boxed{\text{row } A =}$ Col $A^T$ =

$$\begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}^T = \begin{bmatrix} -2 & 2 & -3 \\ 4 & -6 & 8 \\ -2 & -3 & 2 \\ -4 & 1 & -3 \end{bmatrix}$$

$r_2 \leftarrow r_2 + 2r_1$
$r_3 \leftarrow r_3 - r_1$
$r_4 \leftarrow r_4 - 2r_1$

$$\begin{bmatrix} -2 & 2 & -3 \\ 0 & -2 & 2 \\ 0 & -5 & 5 \\ 0 & -3 & 3 \end{bmatrix}$$

$r_3 \leftarrow r_3 - \frac{5}{2} r_2$

$r_4 \leftarrow r_4 - \frac{3}{2} r_2$

echelon form of $A^T$

$$\begin{array}{ccc} P & P & F \end{array}$$
$$\begin{bmatrix} -2 & 2 & -3 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^T = \begin{array}{c} \begin{array}{ccc} P & P & F \end{array} \\ \begin{bmatrix} -2 & 2 & -3 \\ 4 & -6 & 8 \\ -2 & -3 & 2 \\ -4 & 1 & -3 \end{bmatrix} \end{array}$$

Basis for Col $A^T$ is $\left\{ \begin{bmatrix} -2 \\ 4 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ -3 \\ 1 \end{bmatrix} \right\}$

$\|$

basis for row $A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5/2 \\ 3/2 \end{bmatrix} \right\}$

$$\begin{bmatrix} -2 \\ 4 \\ -2 \\ -4 \end{bmatrix} \boxed{\checkmark} = c_1 \begin{bmatrix} 1 \\ 0 \\ 6 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 5/2 \\ 3/2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 0 \\ 6 \\ 5 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 5/2 \\ 3/2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 4 \\ -2\cdot 6 + 4\cdot\frac{5}{2} \\ -2\cdot 5 + 4\cdot 3/2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -12+10 \\ -10+6 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -6 \\ -3 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 6 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 5/2 \\ 3/2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 6 \\ 5 \end{bmatrix} + (-6) \begin{bmatrix} 0 \\ 1 \\ 5/2 \\ 3/2 \end{bmatrix}$$

Idea of writing one basis in term of another.

Let $V \subseteq \mathbb{R}^m$ be a subspace, and

$\mathcal{B} = \{b_1, b_2, \dots, b_n\}$ be a basis of $V$ and

$\mathcal{C} = \{c_1, c_2, \dots, c_n\}$ be another basis of $V$.

note $m \geq n$ so that $\mathcal{B}$ and $\mathcal{C}$ are linearly indep.

Define $B = [b_1 \vdots b_2 \cdots b_n] \in \mathbb{R}^{m\times n}$ and $C = [c_1 \vdots c_2 \cdots c_n] \in \mathbb{R}^{m\times n}$

Since $\mathcal{B}$ spans $V$ then $x \in V$ implies there is some $u \in \mathbb{R}^n$ such that $x = Bu$.

Since $\mathcal{C}$ spans $V$ there is $v \in \mathbb{R}^n$ so $x = Cv$.

By the unique representation theorem given a fixed $x$ there is only one $u$ such that $x = Bu$

only one $v$ such that $x = Cv$.

Therefore $u$ depends on $x$ like a function of $x$.

Notation $u = [x]_{\mathcal{B}} = f(x) \in \mathbb{R}^n$

means some function of $x$

Similarly $v = [x]_{\mathcal{C}} = g(x) \in \mathbb{R}^n$ some function of $x$.

Goal is to find a relationship between $u$ and $v$.

Since $b_i \in V$ then we can solve $b_i = Cv$ for some $v \in \mathbb{R}^n$

notation $v = [b_i]_{\mathcal{C}}$

$c_i \in V$ then we can solve $c_i = Bu$ for some $u \in \mathbb{R}^n$

notation $u = [c_i]_{\mathcal{B}}$

Thus

$$b_i = C[b_i]_{\mathcal{C}} \quad \text{and} \quad c_i = B[c_i]_{\mathcal{B}} \quad \text{for every } i$$

Now

$$B = [b_1 \vdots b_2 \cdots b_n] = \left[ C[b_1]_{\mathcal{C}} \vdots C[b_2]_{\mathcal{C}} \cdots C[b_n]_{\mathcal{C}} \right]$$

$$= C\left[ [b_1]_{\mathcal{C}} \vdots [b_2]_{\mathcal{C}} \cdots [b_n]_{\mathcal{C}} \right]$$

Now

$$x = B[x]_{\mathcal{B}} = C\left[ [b_1]_{\mathcal{C}} \vdots [b_2]_{\mathcal{C}} \cdots [b_n]_{\mathcal{C}} \right][x]_{\mathcal{B}}$$

also

$$x = C[x]_{\mathcal{C}}$$

Thus

$$C\left[ [b_1]_{\mathcal{C}} \vdots [b_2]_{\mathcal{C}} \cdots [b_n]_{\mathcal{C}} \right][x]_{\mathcal{B}} = C[x]_{\mathcal{C}}$$

$$C\left\{ \left[ [b_1]_{\mathcal{C}} \vdots [b_2]_{\mathcal{C}} \cdots [b_n]_{\mathcal{C}} \right][x]_{\mathcal{B}} - [x]_{\mathcal{C}} \right\} = 0.$$

Now $\mathcal{C}$ is a basis, so the columns of $C$ are linearly independent. This means $\text{Nul}\, C = \{0\}$.

$$\left[ [b_1]_{\mathcal{C}} \vdots [b_2]_{\mathcal{C}} \cdots [b_n]_{\mathcal{C}} \right] [x]_{\mathcal{B}} - [x]_{\mathcal{C}} \in \text{Nul}\, C$$

so

$$\underbrace{\left[ [b_1]_{\mathcal{C}} \vdots [b_2]_{\mathcal{C}} \cdots [b_n]_{\mathcal{C}} \right]}_{\substack{\text{Change of basis matrix} \\ \text{from the } \mathcal{B} \text{ basis to the } \mathcal{C} \text{ basis}}} [x]_{\mathcal{B}} = [x]_{\mathcal{C}}$$

Notation

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left[ [b_1]_{\mathcal{C}} \vdots [b_2]_{\mathcal{C}} \cdots [b_n]_{\mathcal{C}} \right] \in \mathbb{R}^{n \times n}$$

If I had written the book I'd have denoted this matrix as $[\mathcal{C} \leftarrow \mathcal{B}]$

By symmetry

$$\left[ [c_1]_{\mathcal{B}} \vdots [c_2]_{\mathcal{B}} \cdots [c_n]_{\mathcal{B}} \right] [x]_{\mathcal{C}} = [x]_{\mathcal{B}}$$

Notation

$$\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \left[ [c_1]_{\mathcal{B}} \vdots [c_2]_{\mathcal{B}} \cdots [c_n]_{\mathcal{B}} \right] \in \mathbb{R}^{n \times n}$$

Note that

invertible matrices

$$\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} \; \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = I \quad \text{and} \quad \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} \; \underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = I$$

What is $[b_i]_{\mathcal{B}} = ?$ and $[c_i]_{\mathcal{E}} = ?$

Answer $[b_i]_{\mathcal{B}} = e_i$ and $[c_i]_{\mathcal{E}} = e_i$

why $B[b_i]_{\mathcal{B}} = b_i$

and. $B e_i = b_i$