1. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space V, and suppose $\mathbf{b}_1 = 6\mathbf{c}_1 - 2\mathbf{c}_2$ and $\mathbf{b}_2 = 9\mathbf{c}_1 - 4\mathbf{c}_2$.

a. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

b. Find $[\mathbf{x}]_{\mathcal{C}}$ for $\mathbf{x} = -3\mathbf{b}_1 + 2\mathbf{b}_2$. Use part (a).

now

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left[[b_1]_{\mathcal{C}} \,\vdots\, [b_2]_{\mathcal{C}} \cdots [b_n]_{\mathcal{C}} \right] \in \mathbb{R}^{n \times n}$$

from $\mathcal{B}$ to $\mathcal{C}$

If I had written the book I'd have denoted this matrix as $[\mathcal{C} \leftarrow \mathcal{B}]$

$$[\mathcal{C} \leftarrow \mathcal{B}] = \left[[b_1]_{\mathcal{C}} \,\vdots\, [b_2]_{\mathcal{C}} \right]$$

$$b_1 = 6c_1 - 2c_2 = \left[c_1 \,\vdots\, c_2 \right] \begin{bmatrix} 6 \\ -2 \end{bmatrix} = C \begin{bmatrix} 6 \\ -2 \end{bmatrix} = C \left[b_1 \right]_{\mathcal{C}}$$

Thus solving $b_1 = Cu$ for u then $u = \begin{bmatrix} 6 \\ -2 \end{bmatrix} = [b_1]_{\mathcal{C}}$

$$b_2 = 9c_1 - 4c_2 = C \begin{bmatrix} 9 \\ -4 \end{bmatrix}$$ and so $[b_2]_{\mathcal{C}} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}$

$$[\mathcal{C} \leftarrow \mathcal{B}] = \left[[b_1]_{\mathcal{C}} \,\vdots\, [b_2]_{\mathcal{C}} \right] = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$$

$$[\mathcal{B} \leftarrow \mathcal{C}] = [\mathcal{C} \leftarrow \mathcal{B}]^{-1} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}^{-1} =$$

Recall inverse formula for a 2x2 matrix

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} \det A_1(e_1) & \det A_1(e_2) & \cdots & \det A_1(e_n) \\ \det A_2(e_1) & \det A_2(e_2) & & \\ \vdots & & \ddots & \\ \det A_n(e_1) & \cdots & & \det A_n(e_n) \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & & \vdots \\ \vdots & & \ddots & \\ C_{1n} & \cdots & & C_{nn} \end{bmatrix}$$

row

$\underbrace{C_{ij}}_{\text{constant}} = (-1)^{i+j} \det A_{ij}$

for $n=2$ $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad$ inverse of a 2x2 matrix...

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

also in Chapter 2.2

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

$$[\mathcal{B} \leftarrow \mathcal{C}] = [\mathcal{C} \leftarrow \mathcal{B}]^{-1} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}^{-1} = \frac{1}{-24+18}\begin{bmatrix} -4 & -9 \\ 2 & 6 \end{bmatrix}$$

$$= \frac{1}{-6}\begin{bmatrix} -4 & -9 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 4/6 & 9/6 \\ -2/6 & -6/6 \end{bmatrix} = \begin{bmatrix} 2/3 & 3/2 \\ -1/3 & -1 \end{bmatrix}$$

b. Find $[\mathbf{x}]_{\mathcal{C}}$ for $\mathbf{x} = -3\mathbf{b}_1 + 2\mathbf{b}_2$. Use part (a).

$$x = \begin{bmatrix} b_1 & \vdots & b_1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \mathcal{B} \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Therefore $[x]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$$[x]_{\mathcal{C}} = [\mathcal{C} \leftarrow \mathcal{B}][x]_{\mathcal{B}} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -18+18 \\ 6-8 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

recall from last time

$$\underbrace{\begin{bmatrix} [b_1]_{\mathcal{C}} & \vdots & [b_2]_{\mathcal{C}} & \cdots & [b_n]_{\mathcal{C}} \end{bmatrix}}_{\text{change of basis matrix}} [x]_{\mathcal{B}} = [x]_{\mathcal{C}}$$

Change of basis matrix from the $\mathcal{B}$ basis to the $\mathcal{C}$ basis $[\mathcal{C} \leftarrow \mathcal{B}]$

Check the answer

$\mathbf{b}_1 = 6\mathbf{c}_1 - 2\mathbf{c}_2$ and $\mathbf{b}_2 = 9\mathbf{c}_1 - 4\mathbf{c}_2$. $\mathbf{x} = -3\mathbf{b}_1 + 2\mathbf{b}_2$.

$$x = -3(6c_1 - 2c_2) + 2(9c_1 - 4c_2) = \cancel{-18c_1} + 6c_2 + \cancel{18c_1} - 8c_2 = -2c_2$$

$$[x]_{\mathcal{C}} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Theorem 12 in chapt 4

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

PROOF If $H = \{\mathbf{0}\}$, then certainly $\dim H = 0 \le \dim V$. Otherwise, let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be any linearly independent set in H. If S spans H, then S is a basis for H. Otherwise, there is some $\mathbf{u}_{k+1}$ in H that is not in Span S. But then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}\}$ will be linearly independent, because no vector in the set can be a linear combination of vectors that precede it (by Theorem 4).

So long as the new set does not span H, we can continue this process of expanding S to a larger linearly independent set in H. But the number of vectors in a linearly independent expansion of S can never exceed the dimension of V, by Theorem 10. So eventually the expansion of S will span H and hence will be a basis for H, and $\dim H \le \dim V$. ■

Look at the proofs of Theorem 11 and 13 over spring break to practice the skill of reading mathematics and try to write your own versions of these arguments filling in details to practice the skill of writing mathematics

understanding stuff.

8. $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$B = \begin{bmatrix} -1 & 1 \\ 8 & -5 \end{bmatrix}$ $\qquad C = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

recall

$$B = C \underbrace{\left[[b_1]_e \; [b_2]_e \; \cdots \; [b_n]_e \right]}_{[e \leftarrow B]}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 8 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} [e \leftarrow B]$$

Therefore

$$[\mathcal{E} \leftarrow \mathcal{B}] = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 1 \\ 8 & -5 \end{bmatrix} = \frac{1}{-3} \begin{bmatrix} 1 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 8 & -5 \end{bmatrix}$$

$$= \frac{-1}{3} \begin{bmatrix} -9 & 6 \\ 12 & -9 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 8 & -5 \end{bmatrix}$$

$$[\mathcal{B} \leftarrow \mathcal{E}] = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}^{-1} \quad \text{finish...}$$

$$\begin{bmatrix} 1 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} -9 & 6 \\ 12 & -9 \end{bmatrix}$$