$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

$\det(A - \lambda I) = (1-\lambda)(4-\lambda)(6-\lambda)$

eigenvalues $\lambda = 1$, 4 and 6.

$\lambda_1 = 1$ $\quad \mathrm{Nul}(A - I) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ $\quad$ eigenvector $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda = 4$ $\quad \mathrm{Nul}(A - 4I) = \mathrm{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}$ $\quad$ " $\quad v_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

$\lambda = 6$ $\quad \mathrm{Nul}(A - 6I) = \mathrm{span}\left\{ \begin{bmatrix} 16 \\ 25 \\ 10 \end{bmatrix} \right\}$ $\quad$ " $\quad v_3 = \begin{bmatrix} 16 \\ 25 \\ 10 \end{bmatrix}$

$\beta = \{e_1, e_2, e_3\}$ $\qquad \mathcal{C} = \{v_1, v_2, v_3\}$

eigenbasis ... basis made of eigenvectors.

$$[\beta \leftarrow \mathcal{C}] = \begin{bmatrix} 1 & 2 & 16 \\ 0 & 3 & 25 \\ 0 & 0 & 10 \end{bmatrix} = [v_1 | v_2 | v_3]$$

$$[\mathcal{C} \leftarrow \beta] = \begin{bmatrix} 1 & 2 & 16 \\ 0 & 3 & 25 \\ 0 & 0 & 10 \end{bmatrix}^{-1}$$

Let's find the inverse (use augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 16 & 1 & 0 & 0 \\ 0 & 3 & 25 & 0 & 1 & 0 \\ 0 & 0 & 10 & 0 & 0 & 1 \end{array}\right]$$

$r_2 \leftarrow r_2 - \frac{25}{10} r_3 = r_2 - \frac{5}{2} r_3$

$r_1 \leftarrow r_1 - \frac{16}{10} r_3 = r_1 - \frac{8}{5} r_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -8/5 \\ 0 & 3 & 0 & 0 & 1 & -5/2 \\ 0 & 0 & 10 & 0 & 0 & 1 \end{array}\right]$$

$r_1 \leftarrow r_1 - \frac{2}{3} r_2$

$-\frac{8}{5} - \frac{2}{3}\left(-\frac{5}{2}\right) = \frac{-24}{15} + \frac{25}{15} = 1/15$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2/3 & 1/15 \\ 0 & 3 & 0 & 0 & 1 & -5/2 \\ 0 & 0 & 10 & 0 & 0 & 1 \end{array}\right]$$

$r_2 \leftarrow \frac{1}{3} r_2$

$r_3 \leftarrow \frac{1}{10} r_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2/3 & 1/15 \\ 0 & 1 & 0 & 0 & 1/3 & -5/6 \\ 0 & 0 & 1 & 0 & 0 & 1/10 \end{array}\right]$$

The representation of A with respect to the eigenbasis $\mathcal{E}$.

$$[\mathcal{E} \leftarrow \mathcal{B}] A [\mathcal{B} \leftarrow \mathcal{E}] = \begin{bmatrix} 1 & -2/3 & 1/15 \\ 0 & 1/3 & -5/6 \\ 0 & 0 & 1/10 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 2 & 16 \\ 0 & 3 & 25 \\ 0 & 0 & 10 \end{bmatrix}}_{[v_1 | v_2 | v_3]}$$

$$\begin{array}{r} 16 \\ 50 \\ 30 \\ \hline 96 \end{array} \qquad \begin{array}{r} 100 \\ 50 \\ \hline 150 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 16 \\ 0 & 3 & 25 \\ 0 & 0 & 10 \end{bmatrix}$$

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$v_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

$v_3 = \begin{bmatrix} 16 \\ 25 \\ 10 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 8 & 96 \\ 0 & 12 & 150 \\ 0 & 0 & 60 \end{bmatrix}$$

easier way

$$A [v_1 | v_2 | v_3] = [Av_1 | Av_2 | Av_3] = [\lambda_1 v_1 \ \ \lambda_2 v_2 \ \ \lambda_3 v_3]$$

$$= \left[1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \middle| \ 4\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \middle| \ 6\begin{bmatrix} 16 \\ 25 \\ 10 \end{bmatrix} \right] = \begin{bmatrix} 1 & 8 & 96 \\ 0 & 12 & 150 \\ 0 & 0 & 60 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 8 & 96 \\ 0 & 12 & 150 \\ 0 & 0 & 60 \end{bmatrix}$$

$8 - \frac{2}{3} 12 = 8 - 8 = 0$

$96 - \frac{2}{3} 150 + \frac{1}{15} 60 = 96 - 100 + 4 = 0$

$\frac{1}{3} 150 - \frac{5}{6} 60 = 50 - 50 = 0$

$$\begin{bmatrix} 1 & -2/3 & 1/15 \\ 0 & 1/3 & -5/6 \\ 0 & 0 & 1/10 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

① The zeros in the upper right were not by accident

② The diagonal is the eigenvalues.

$$[e \leftarrow \beta] A \cdot \underbrace{[\beta \leftarrow e]}_{P = [v_1 \vdots v_2 \vdots v_3]} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}}_{D}$$

Thus $P^{-1} A P = D$

If there is an eigenbasis of eigenvectors of $A \in \mathbb{R}^{n \times n}$

Then

$$P^{-1} A P = D \quad \text{where } P = [v_1 \vdots v_2 \vdots \cdots \vdots v_n]$$

$$\text{and } D = \begin{bmatrix} \lambda_1 & & \bigcirc \\ & \lambda_2 & \\ & & \ddots & \\ \bigcirc & & & \lambda_n \end{bmatrix}$$

Here $A v_i = \lambda_i v_i$ for $i = 1, \ldots, n$

$$AP = A\left[v_1 \vdots v_2 \vdots \cdots \vdots v_n\right] = \left[Av_1 \vdots Av_2 \vdots \cdots \vdots Av_n\right]$$

$$= \left[\lambda_1 v_1 \vdots \lambda_2 v_2 \vdots \cdots \vdots \lambda_n v_n\right]$$

$$P\begin{bmatrix}\lambda_1\\0\\\vdots\\0\end{bmatrix} = \left[v_1 \vdots v_2 \vdots \cdots \vdots v_n\right]\begin{bmatrix}\lambda_1\\0\\\vdots\\0\end{bmatrix} = \lambda_1 v_1 + 0v_2 + \cdots + 0v_n = \lambda_1 v_1$$

$$P\begin{bmatrix}0\\\lambda_2\\\vdots\\0\end{bmatrix} = \left[v_1 \vdots v_2 \vdots \cdots \vdots v_n\right]\begin{bmatrix}0\\\lambda_2\\\vdots\\0\end{bmatrix} = 0v_1 + \lambda_2 v_2 + 0v_3 + \cdots + 0v_n = \lambda_2 v_2$$

Therefore

$$\left[\lambda_1 v_1 \vdots \lambda_2 v_2 \vdots \cdots \vdots \lambda_n v_n\right] = \left[P\begin{bmatrix}\lambda_1\\0\\\vdots\\0\end{bmatrix} \vdots P\begin{bmatrix}0\\\lambda_2\\\vdots\\0\end{bmatrix} \vdots \cdots \vdots P\begin{bmatrix}0\\0\\\vdots\\\lambda_n\end{bmatrix}\right]$$

$$= P\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\0 & \lambda_2 & \cdots & 0\\\vdots & \vdots & & \vdots\\0 & 0 & \cdots & \lambda_n\end{bmatrix} = PD$$

Therefore in general

similarity transformation
or change of basis...

$$AP = PD \quad \text{or} \quad P^{-1}AP = D \quad \text{or} \quad \underbrace{A = PDP^{-1}}_{}$$

This is a factorization
of the matrix A.

Example from before

$$\underbrace{\begin{bmatrix} 1 & -2/3 & 1/15 \\ 0 & 1/3 & -5/6 \\ 0 & 0 & 1/10 \end{bmatrix}}_{P^{-1}} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 2 & 16 \\ 0 & 3 & 25 \\ 0 & 0 & 10 \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}}_{D}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}}_{A} \underset{=}{=} \underbrace{\begin{bmatrix} 1 & 2 & 16 \\ 0 & 3 & 25 \\ 0 & 0 & 10 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -2/3 & 1/15 \\ 0 & 1/3 & -5/6 \\ 0 & 0 & 1/10 \end{bmatrix}}_{P^{-1}}$$

Want to find a matrix such that if I square it the result equal A.

What about D?

$$D^{1/2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}^{1/2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{6} \end{bmatrix}$$

Since:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$A = PDP^{-1} = PD^{1/2}D^{1/2}P^{-1} = PD^{1/2}\,I\,D^{1/2}P^{-1}$$

$$= PD^{1/2}P^{-1}PD^{1/2}P^{-1} = \underbrace{(PD^{1/2}P^{-1})}_{A^{1/2}}\underbrace{(PD^{1/2}P^{-1})}_{A^{1/2}}$$

Therefore $A^{1/2} = PD^{1/2}P^{-1}$

Note there are many square roots

$$\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 2 & 0 \\ 0 & 0 & \pm\sqrt{6} \end{bmatrix}$$

$2 \cdot 2 \cdot 2 = 8$ different square roots.