**7.** The matrix $A$ given by

$$A = \begin{bmatrix} 12.6 & 2.2 & 8.6 \\ 9.6 & 3.2 & 6.6 \\ -16.8 & -3.6 & -11.8 \end{bmatrix}$$

has eigenvalues $\lambda_i$ and eigenvectors $x_i$ given by

means $Ax_i = \lambda_i x_i$ for $i = 1, 2, 3$

$$\lambda_1 = 2, \quad x_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \qquad \lambda_2 = 3, \quad x_2 = \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}, \qquad \lambda_3 = -1, \quad x_3 = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}.$$

Find an invertible matrix $S$ and a diagonal matrix $D$ such that $A = SDS^{-1}$.

**(i)** What is $D$?

$$S = \left[ x_1 \,\vdots\, x_2 \,\vdots\, x_3 \right]$$

change of basis matrix from the eigenbasis to the standard basis

$$AS = A\left[ x_1 \,\vdots\, x_2 \,\vdots\, x_3 \right] = \left[ Ax_1 \,\vdots\, Ax_2 \,\vdots\, Ax_3 \right]$$

$$= \left[ \lambda_1 x_1 \,\vdots\, \lambda_2 x_2 \,\vdots\, \lambda_3 x_3 \right] = \left[ x_1 \,\vdots\, x_2 \,\vdots\, x_3 \right] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = SD$$

$A = SDS^{-1}$ where

$$S = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \\ -2 & -3 & -5 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

**8.** Find the eigenvalues and eigenvectors of the matrix $A$ where

$$A = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}.$$

Trying to solve $Ax = \lambda x$ for $x$ and $\lambda$.

$$(A - \lambda I)x = 0$$

Since $x \neq 0$ to be an eigenvector then $\mathrm{Nul}(A - \lambda I)$ has to be non-trivial. So $A - \lambda I$ has free variables so $\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \det \begin{bmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{bmatrix} = (-3-\lambda)(-2-\lambda) - 2$$

$$= (\lambda + 3)(\lambda + 2) - 2 = \lambda^2 + 5\lambda + 6 - 2 = \underbrace{\lambda^2 + 5\lambda + 4}$$

could happen that this quadratic doesn't factor using rational factors...

$$= (\lambda + 4)(\lambda + 1) = 0$$

So eigenvalues are $\lambda = -4, -1$

$\lambda = -4$

$$\mathrm{Nul}(A - \lambda I) = \mathrm{Nul} \begin{bmatrix} -3-(-4) & 2 \\ 1 & -2-(-4) \end{bmatrix} = \mathrm{Nul} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

P F

$x_1 + 2x_2 = 0$ so $x_1 = -2x_2$

Solutions $\mathrm{Nul} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2 : x_2 \in \mathbb{R} \right\}$

an eigenvector is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ there are others.

$\lambda = -1$

$$\mathrm{Nul}(A - \lambda I) = \mathrm{Nul} \begin{bmatrix} -3-(-1) & 2 \\ 1 & -2-(-1) \end{bmatrix} = \mathrm{Nul} \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$$

second row is equal $-\frac{1}{2}$ times the first.

$x_1 - x_2 = 0$ $\quad$ $x_1 = x_2$ $\quad$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$

Solutions $\quad \mathrm{Nul} \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2 : x_2 \in \mathbb{R} \right\}$

an eigenvector for $\lambda = -1$.

Eigenvalue - eigenvector pairs are

$\lambda_1 = -4, \ x_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\lambda_2 = -1, \ x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}$

$A = PDP^{-1}$

$$\underset{A}{\begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}} \cdot \underset{P}{\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}} = \underbrace{\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}}_{P} \underset{D}{\begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}}$$

Cramer's rule for $n=2$ is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$P^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{-3} \begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

$$\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix} = 1 \cdot 5 \cdot 8 \cdot 10$$

$A = PDP^{-1}$

$$\begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

Can still take a square root

$$\begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}^{1/2} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \pm 2i & 0 \\ 0 & \pm i \end{bmatrix} \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

What about powers?

$$\begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}^{2} = A^2 = AA = PD\cancel{P^{-1}}\,\cancel{P}DP^{-1} = PD^2P^{-1}$$

$$= P \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}^2 P^{-1} = P \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix} P^{-1}$$

What about polynomials?

$$p(t) = at^2 + bt + c$$

$$p\left(\begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}\right) = a\begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}^2 + b\begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= P \begin{bmatrix} p(-4) & 0 \\ 0 & p(-1) \end{bmatrix} P^{-1}$$

Continue on Wednesday...