2. Consider the matrix A with reduced row eschelon form R where

$$A = \begin{array}{c} \begin{array}{ccccc} P & F & P & P & F \end{array} \\ \begin{bmatrix} -3 & 6 & 0 & 1 & 3 \\ -2 & 4 & 3 & 2 & 5 \\ -2 & 4 & 1 & 1 & 4 \end{bmatrix} \end{array} \quad \text{and} \quad R = \begin{array}{c} \begin{array}{ccccc} P & F & P & P & F \end{array} \\ \begin{bmatrix} \boxed{1} & -2 & 0 & 0 & -4 \\ 0 & 0 & \boxed{1} & 0 & 5 \\ 0 & 0 & 0 & \boxed{1} & -9 \end{bmatrix} \end{array}.$$

pivots

Find. Col A and Nul A

A basis for Col A is $\left\{ \begin{bmatrix} -3 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

Note solving $Ax=0$ is the same as $Rx=0$.

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} \boxed{1} & -2 & 0 & 0 & -4 \\ 0 & 0 & \boxed{1} & 0 & 5 \\ 0 & 0 & 0 & \boxed{1} & -9 \end{bmatrix} \end{array}$$

$$x_1 - 2x_2 - 4x_5 = 0$$
$$x_3 + 5x_5 = 0$$
$$x_4 - 9x_5 = 0$$

Solve for pivot variables in terms of the free variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + 4x_5 \\ x_2 \\ -5x_5 \\ 9x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 4 \\ 0 \\ -5 \\ 9 \\ 1 \end{bmatrix} x_5$$

Basis for Nul A is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 9 \\ 1 \end{bmatrix} \right\}.$

3. Answer the following true false questions:

(i) If A is invertible then $(\det(A^{-1}))(\det A) = 1$.

(A) True 14

(B) False 0

Why $\det A \det B = \det(AB)$

so $\det(A^{-1}) \det A = \det(A^{-1}A) = \det I = 1$

Another T/F question:

(ib). $\det((A^{-1})(\det A)) = 1$

(A) True 0

(B) False 5

(ii) If A is invertible, then 0 is not an eigenvalue of A.

(A) True 7

(B) False 0

Why. If 0 is an eigenvalue then $\text{Nul}(A - 0I)$ is non-trivial so $\det A = 0$ so A is not invertible.

why.

(iii) If $A \in \mathbf{R}^{n\times n}$ is triangular, then $\det A$ is the product of the diagonal entries of A.

(A) True

(B) False

(iv) The matrices A and $B^{-1}AB$ have the same sets of eigenvalues for every invertible matrix B.

(A) True

(B) False

4. Let λ be an eigenvalue of an invertible matrix A. Show λ^{-1} is an eigenvalue of A^{-1}.

$$\det((A^{-1})\underbrace{(\det A)}_{\alpha}) = (\det A)^n \det(A^{-1}) = (\det A)^{n-1} \underbrace{\det A \det(A^{-1})}_{=1}$$

$$= (\det A)^{n-1}$$

If $A \in \mathbb{R}^n$ then $\det(\alpha A) = \alpha^n \det A$

$$\det(\alpha I) = \det\left(\alpha \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}\right) = \det \begin{bmatrix} \alpha & & & \\ & \alpha & & \\ & & \ddots & \\ & & & \alpha \end{bmatrix} = \alpha \cdot \alpha \cdot \alpha \cdots \alpha = \alpha^n$$

n 1's on the diagonal.

$$\det(\alpha A) = \det(\alpha I A) = \det(\alpha I)\det A = \alpha^n \det A$$

(iv) The matrices A and $B^{-1}AB$ have the same sets of eigenvalues for every invertible matrix B.

Let λ be an eigenvalue of A. That means $\text{Nul}(A-\lambda I)$ is non-trivial so there is an eigenvector $x \neq 0$ with $x \in \text{Nul}(A-\lambda I)$ such that

$$Ax = \lambda x$$

Let $y = B^{-1}x$ then $By = BB^{-1}x = x$

Then $B^{-1}ABy = B^{-1}Ax = B^{-1}\lambda x = \lambda B^{-1}x = \lambda y$

Thus λ is an eigenvalue of the eigenvector y for $B^{-1}AB$

The above argument is reversible so the eigenvalues of $B^{-1}AB$ are also eigenvalues of A.

4. Let λ be an eigenvalue of an invertible matrix A. Show λ^{-1} is an eigenvalue of A^{-1}.

Let x be the eigenvector of A corresponding to λ.

Then

$$Ax = \lambda x.$$

Now multiply by the inverse

$$A^{-1}Ax = A^{-1}\lambda x,$$

$$x = \lambda A^{-1}x$$

Therefore $A^{-1}x = \frac{1}{\lambda}x$

which implies $\frac{1}{\lambda}$ is an eigenvalue of the matrix A^{-1}.

Note that since A is invertible then 0 is not an eigenvalue of A.

$A = PDP^{-1}$

$$\begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

Applications: $A \in \mathbb{R}^{n \times n}$ maybe $n=2$

Evaluating a polynomial with a matrix as the input:

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_m t^m$$

poly of degree m

$$p(A) = a_0 \underset{m \times m}{I} + a_1 \underset{m \times m}{A} + a_2 \underset{m \times m}{A^2} + \cdots + a_m \underset{m \times m}{A^m}$$

simplify this

$$= a_0 I + a_1 PDP^{-1} + a_2 (PDP^{-1})^2 + \cdots + a_m (PDP^{-1})^m$$

$$(PDP^{-1})^2 = PD\cancel{P^{-1}}\cancel{P}DP^{-1} = PD^2P^{-1}$$

$$(PDP^{-1})^3 = PD\cancel{P^{-1}}\cancel{P}D\cancel{P^{-1}}\cancel{P}DP^{-1} = PD^3P^{-1}$$

$\vdots$

$$(PDP^{-1})^m = PD^mP^{-1}$$

$$p(A) = a_0 PP^{-1} + a_1 PDP^{-1} + a_2 PD^2P^{-1} + \cdots + a_m PD^mP^{-1}$$

$n=2$

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad D^2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$$

$$D^3 = D^2 D = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{bmatrix}$$

$$\vdots$$

$$D^m = \begin{bmatrix} \lambda_1^m & 0 \\ 0 & \lambda_2^m \end{bmatrix}$$

$$p(A) = a_0 P P^{-1} + a_1 P D P^{-1} + a_2 P D^2 P^{-1} + \cdots + a_m P D^m P^{-1}$$

$$= P \left(a_0 I + a_1 D + a_2 D^2 + \cdots a_m D^m \right) P^{-1}$$

$$= P \left(\underbrace{a_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_1 \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} + a_2 \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} + \cdots + a_m \begin{bmatrix} \lambda_1^m & 0 \\ 0 & \lambda_2^m \end{bmatrix}}_{\text{add up}} \right) P^{-1}$$

$$= P \begin{bmatrix} a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \cdots + a_m\lambda_1^m & 0 \\ 0 & a_0 + a_1\lambda_2 + a_2\lambda_2^2 + \cdots + a_m\lambda_2^m \end{bmatrix} P^{-1}$$

$$= P \begin{bmatrix} p(\lambda_1) & 0 \\ 0 & p(\lambda_2) \end{bmatrix} P^{-1}$$