$$\underbrace{\begin{bmatrix}-3 & 2\\ 1 & -2\end{bmatrix}}_{A} \underbrace{=}_{=} \underbrace{\overbrace{\begin{bmatrix}-2 & 1\\ 1 & 1\end{bmatrix}}^{\text{eigenvectors}}}_{P} \underbrace{\overbrace{\begin{bmatrix}-4 & 0\\ 0 & -1\end{bmatrix}}^{\text{eigenvalues}}}_{D} \underbrace{\begin{bmatrix}-1/3 & 1/3\\ 1/3 & 2/3\end{bmatrix}}_{P^{-1}}$$

We know that

$\lambda_1 = -4, \quad v_1 = \begin{bmatrix}-2\\ 1\end{bmatrix}$ satisfies $Av_1 = \lambda_1 v_1$

$\lambda_1 = -1, \quad v_2 = \begin{bmatrix}1\\ 1\end{bmatrix}$ satisfies $Av_2 = \lambda_2 v_2$

If $A = PDP^{-1}$

$$A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1} = P\begin{bmatrix}\lambda_1^2 & 0\\ 0 & \lambda_2^2\end{bmatrix}P^{-1}$$

$\vdots$

$$A^m = PD^mP^{-1} = P\begin{bmatrix}\lambda_1^m & 0\\ 0 & \lambda_2^m\end{bmatrix}P^{-1}$$

Evaluating a polynomial at a matrix $A \in \mathbb{R}^{n\times n}$

$$p(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_m t^m$$ ← general polynomial.

$$p(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_m t^m$$ ← definition of $p(A)$

Simplify using eigenvectors and eigenvalues

$$p(t) = at^2 + bt + c = \alpha_0 + \alpha_1 t + \alpha_2 t^2$$

where $m = 2$, $\alpha_0 = c$, $\alpha_1 = b$ and $\alpha_2 = a$

$$p(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_m t^m$$

$$= \alpha_0 PIP^{-1} + \alpha_1 PDP^{-1} + \alpha_2 PD^2P^{-1} + \cdots + \alpha_m PD^mP^{-1}$$

$$= P\left(\alpha_0 I + \alpha_1 D + \alpha_2 D^2 + \cdots + \alpha_m D^m\right)P^{-1}$$

If $A \in \mathbb{R}^{n \times n}$ and $A = PDP^{-1}$ where P is the matrix of eigenvectors and D the matrix of eigenvalues...

Then $P = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ and $D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$

$$D^2 = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & & & 0 \\ & \lambda_2^2 & & \\ & & \ddots & \\ 0 & & & \lambda_n^2 \end{bmatrix}$$

$$\vdots$$

$$D^m = \begin{bmatrix} \lambda_1^m & & & 0 \\ & \lambda_2^m & & \\ & & \ddots & \\ 0 & & & \lambda_n^m \end{bmatrix}$$

Using an eigenbasis leads to a diagonal matrix D. Matrix powers of D turn into powers of the scalar eigenvalues.

$$p(A) = P\left(\alpha_0 I + \alpha_1 D + \alpha_2 D^2 + \cdots + \alpha_m D^m\right)P^{-1}$$

$$= P\Bigg(\underbrace{\alpha_0\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} + \alpha_1\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} + \alpha_2\begin{bmatrix} \lambda_1^2 & & & 0 \\ & \lambda_2^2 & & \\ & & \ddots & \\ 0 & & & \lambda_n^2 \end{bmatrix} + \cdots + \alpha_m\begin{bmatrix} \lambda_1^m & & & 0 \\ & \lambda_2^m & & \\ & & \ddots & \\ 0 & & & \lambda_n^m \end{bmatrix}}_{\text{add this up}}\Bigg)P^{-1}$$

$$= P\left(\begin{bmatrix} \alpha_0 & & 0 \\ & \alpha_0 & \\ 0 & & \alpha_0 \end{bmatrix} + \begin{bmatrix} \alpha_1\lambda_1 & & & 0 \\ & \alpha_1\lambda_2 & & \\ & & \ddots & \\ 0 & & & \alpha_1\lambda_n \end{bmatrix} + \begin{bmatrix} \alpha_2\lambda_1^2 & & & 0 \\ & \alpha_2\lambda_2^2 & & \\ & & \ddots & \\ 0 & & & \alpha_2\lambda_n^2 \end{bmatrix} + \cdots + \begin{bmatrix} \alpha_m\lambda_1^m & & & 0 \\ & \alpha_m\lambda_2^m & & \\ & & \ddots & \\ 0 & & & \alpha_m\lambda_n^m \end{bmatrix}\right)P^{-1}$$

$$= P\begin{bmatrix} \alpha_0 + \alpha_1\lambda_1 + \alpha_2\lambda_1^2 + \cdots + \alpha_m\lambda_1^m & & & 0 \\ & \alpha_0 + \alpha_1\lambda_2 + \alpha_2\lambda_2^2 + \cdots + \alpha_m\lambda_2^m & & \\ & & \ddots & \\ 0 & & & \alpha_0 + \alpha_1\lambda_n + \alpha_2\lambda_n^2 + \cdots \alpha_m\lambda_n^m \end{bmatrix}P^{-1}$$

$$= P\begin{bmatrix} p(\lambda_1) & & & 0 \\ & p(\lambda_2) & & \\ & & \ddots & \\ 0 & & & p(\lambda_n) \end{bmatrix}P^{-1}$$

Let consider Taylor polynomials ... For example.

$$e^t = \lim_{m\to\infty} T_m(t) \quad \text{where } T_m(t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^2 + \dots + \frac{1}{m!}t^m$$

We sometimes write

$$e^t = 1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^2 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!}t^k$$

We know how to evaluate a polynomial at a matrix

$$T_m(A) = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots + \frac{1}{m!}A^m$$

$$= P \begin{bmatrix} T_m(\lambda_1) & & & \\ & T_m(\lambda_2) & & 0 \\ & 0 & \ddots & \\ & & & T_m(\lambda_n) \end{bmatrix} P^{-1}$$

Now take limit as $m \to \infty$

$$\lim_{m\to\infty} T_m(A) = \lim_{m\to\infty} P \begin{bmatrix} T_m(\lambda_1) & & & \\ & T_m(\lambda_2) & & 0 \\ & 0 & \ddots & \\ & & & T_m(\lambda_n) \end{bmatrix} P^{-1}$$

matrices represent linear functions and linear functions are continuous.

$$= P \left(\lim_{m\to\infty} \begin{bmatrix} T_m(\lambda_1) & & & \\ & T_m(\lambda_2) & & 0 \\ & 0 & \ddots & \\ & & & T_m(\lambda_n) \end{bmatrix} \right) P^{-1}$$

$$= P \begin{bmatrix} \lim_{m\to\infty} T_m(\lambda_1) & & & \\ & \lim_{m\to\infty} T_m(\lambda) & & 0 \\ 0 & & \ddots & \\ & & & \lim_{m\to\infty} T_m(\lambda_n) \end{bmatrix} P^{-1}$$

Since T_m are Taylor polynomials for e^t and by Taylors theorem they converge to e^t, then

$$\lim_{m\to\infty} T_m(\lambda_i) = e^{\lambda_i} \quad \text{for } i=1,\dots,n.$$

Therefore

$$\lim_{m\to\infty} T_m(A) = P\begin{bmatrix} e^{\lambda_1} & & & 0 \\ & e^{\lambda_2} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n}\end{bmatrix}P^{-1}$$

Note, since the $\lim_{m\to\infty} T_m(\lambda_i)$ existed for every λ_i then the $\lim_{m\to\infty} T_m(A)$ also exists

In this case

$$e^A = P\begin{bmatrix} e^{\lambda_1} & & & 0 \\ & e^{\lambda_2} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n}\end{bmatrix}P^{-1}$$

application Math 285 ODE

Homogeneous. system of linear ODES

$$\begin{aligned}
\dot{y}_1 &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \cancel{+ f_1} \\
\dot{y}_2 &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \cancel{+ f_2} \\
&\vdots \\
\dot{y}_n &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n \cancel{+ f_n}
\end{aligned}$$

$$\dot{y} = Ay \qquad\qquad y' = 3y$$

$$y(t) = \underbrace{e^{At}}_{\text{matric}}\ \underbrace{c}_{\text{vector of constants}} \qquad\qquad y(t) = ce^{3t}$$

For any function that has a convergent Taylor series you can do the same thing

$$\sin(A) = P\begin{bmatrix} \sin\lambda_1 & & & 0 \\ & \sin\lambda_2 & & \\ & & \ddots & \\ 0 & & & \sin\lambda_n\end{bmatrix}P^{-1}$$

$$\cos(A) = P \begin{bmatrix} \cos\lambda_1 & & & 0 \\ & \cos\lambda_2 & & \\ & & \ddots & \\ 0 & & & \cos\lambda_n \end{bmatrix} P^{-1}$$

What about ln? The Taylor series for ln t doesn't converge for all values of t...

r $\ln(1+x)$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$.

$$\ln(\underbrace{1+x}_{t}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

Convergen for $x \in (-1, 1)$.

$t = 1+x \qquad x = t-1$

$$\ln t = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (t-1)^n$$

converges for $t \in (0,2)$

Therefore if $\lambda_i \in (0,2)$ then

$$\ln A = P \begin{bmatrix} \ln\lambda_1 & & \\ & \ln\lambda_2 & \\ & & \ddots \\ & & & \ln\lambda_n \end{bmatrix} P^{-1}$$ is okay