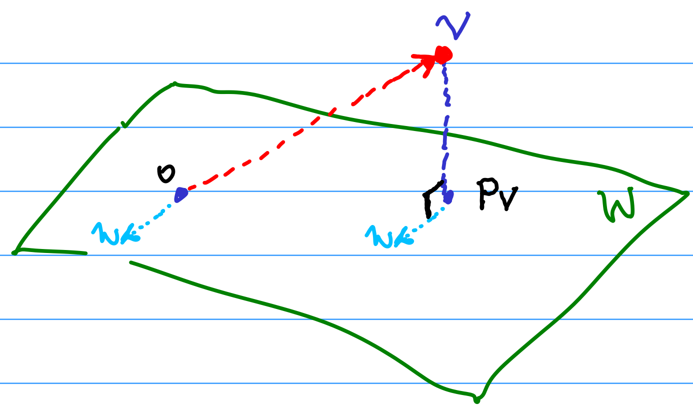
Orthogonal Project

W is a subspace of $\mathbb{R}^n$



$\mathbb{R}^n$

Idea find closest point in W to v

$w \cdot (v - P_v) = 0$

for all $w \in W$.

Suppose $W = \text{span}\{u_1, u_2, \ldots, u_p\}$ where $\{u_1, u_2, \ldots, u_p\}$ is an orthonormal basis;

Thus $u_i \cdot u_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$$U = \left[u_1 \,\vdots\, u_2 \,\vdots\, \cdots \,\vdots\, u_p \right]$$

Then $W = \text{Col}\, U$. $\quad \underset{p \times n}{U^T} \underset{n \times p}{U} = I \in \mathbb{R}^{p \times p}$ $\quad \boxed{P = UU^T \in \mathbb{R}^{n \times n}}$

also $P = P_{u_1} + P_{u_2} + \cdots + P_{u_p}$

Suppose $W = \text{span}\{u_1, u_2, \ldots, u_p\}$ where $\{u_1, u_2, \ldots, u_p\}$ is an orthogonal basis;

Thus $u_i \cdot u_j = \begin{cases} \|u_i\|^2 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ and $\|u_i\| > 0$

$$U = \left[u_1 \,\vdots\, u_2 \,\vdots\, \cdots \,\vdots\, u_p \right]$$

Then $W = \text{Col}\, U$. $\quad U^T U = \begin{bmatrix} \|u_1\|^2 & & & \\ & \|u_2\|^2 & & 0 \\ & 0 & \ddots & \\ & & & \|u_p\|^2 \end{bmatrix} \in \mathbb{R}^{p \times p}$

Let $U = \begin{bmatrix} u_1 & u_2 & \cdots & u_p \end{bmatrix}$ $\quad D = \begin{bmatrix} 1/\|u_1\| & & 0 \\ & 1/\|u_2\| & \\ 0 & & \ddots & \\ & & & 1/\|u_p\| \end{bmatrix}$

columns are orthogonal but not unit vectors

$UD = \begin{bmatrix} \frac{u_1}{\|u_1\|} & \frac{u_2}{\|u_2\|} & \cdots & \frac{u_p}{\|u_p\|} \end{bmatrix}$ ← orthogonal matrix
columns are orthonormal.

$$P = UD(UD)^T = UDD^TU^T = UD^2U^T$$

same

$$U^TU = \begin{bmatrix} \|u_1\|^2 & & 0 \\ & \|u_2\|^2 & \\ 0 & & \ddots & \\ & & & \|u_p\|^2 \end{bmatrix} \qquad (U^TU)^{-1} = \begin{bmatrix} 1/\|u_1\|^2 & & 0 \\ & 1/\|u_2\|^2 & \\ 0 & & \ddots & \\ & & & 1/\|u_p\|^2 \end{bmatrix}$$

Rewrite P in terms of U matrix

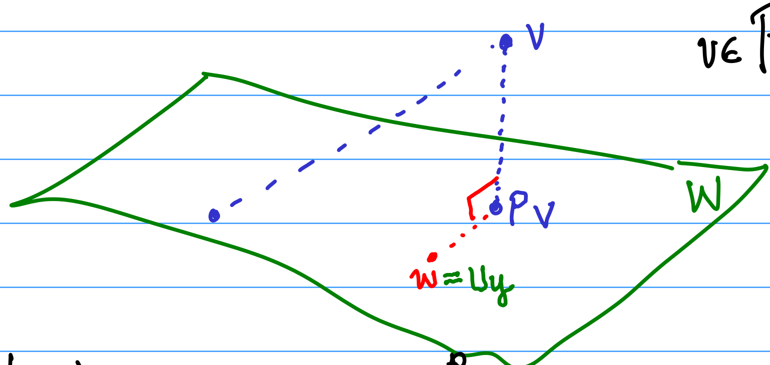
$$P = U(U^TU)^{-1}U^T$$

Suppose $W = \text{span}\{u_1, u_2, \dots, u_p\}$ where $\{u_1, u_2, \dots, u_p\}$ is an basis;

$$U = \begin{bmatrix} u_1 & u_2 & \cdots & u_p \end{bmatrix}$$

$W = \text{Col}\, U \subseteq \mathbb{R}^n$

$\dim W = p$



$v \in \mathbb{R}^n$

want this
$w \cdot (v - Pv) = 0$

Since $w \in W$ then there is $y \in \mathbb{R}^p$ such that $w = Uy$.

Claim $P = U(U^TU)^{-1}U^T$ still works

Check this.

$$Uy \cdot (v - U(U^TU)^{-1}U^Tv) = (Uy)^T(v - U(U^TU)^{-1}U^Tv)$$

$$= y^TU^T(v - U(U^TU)^{-1}U^Tv)$$

$$= y^T(U^Tv - \cancel{U^TU}\underbrace{(\cancel{U^TU})^{-1}}U^Tv) = y^T(\underbrace{U^Tv - U^Tv}_{=0}) = 0$$

But is U^TU really invertible?

Claim $\text{Nul}(U^TU) = \{0\}$. Since U^TU is square, this is enough to conclude U^TU is invertible...

I know $U \in \mathbb{R}^{n \times p}$ has linearly independent columns.

$$Ux = \begin{bmatrix} u_1 & u_2 & \cdots & u_p \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = u_1x_1 + u_2x_2 + \cdots + u_px_p = 0$$

then $x = 0$ because that the definition of linear independence

$\text{Nul}\, U = \{0\}$

Suppose $U^TUx = \vec{0}$ Claim that $x = 0$

I put in a dot product to make algebraic sense of U^T.

$\rightarrow x \cdot U^TUx = 0$, $x^TU^TUx = 0$, $(Ux)^TUx = 0$

$Ux \cdot Ux = 0$, $\|Ux\|^2 = 0$, $Ux = 0$, $x = 0$

Therefore U^TU is invertible and $P = U(U^TU)^{-1}U^T$ makes sense when $\{u_1, u_2, \dots, u_p\}$ is any basis for W.

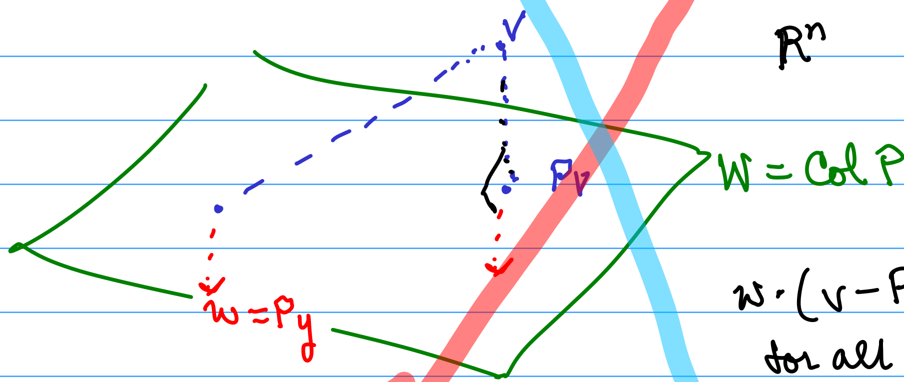
What is an orthogonal projection?

a$^{n\times n}$ matrix such that $P^2=P$ and $\boxed{P^T=P}$

then P is an orthogonal projection onto its column space.

Why?

note $P \in \mathbb{R}^{n\times n}$ and P is not invertible when $\dim W < n$.



$w \cdot (v - Pv) = 0$ for all w

$$Py \cdot (v - Pv) = (Py)^T (v - Pv) = y^T P^T (v - Pv)$$

$$= y^T (P^T v - P^T P v) = y^T (Pv - P^2 v) = y^T (Pv - Pv) = 0$$

Now that

$$P = U(U^TU)^{-1}U^T$$

satisfies $P^2 = P$ and $P^T = P$.