Gram-Schmidt

$\|v_i\|$

$p=3$

$v_1 = u_1$ $\qquad q_1 = \dfrac{v_1}{\|v_1\|}$

$v_2 = u_2 - (q_1 \cdot u_2)\, q_1$ $\qquad q_2 = \dfrac{v_2}{\|v_2\|}$

$v_3 = u_3 - (q_1 \cdot u_3)\, q_1 - (q_2 \cdot u_3)\, q_2$ $\qquad q_3 = \dfrac{v_3}{\|v_3\|}$

$\vdots$

$$\underbrace{\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}}_{n\times 3} = \underbrace{\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}}_{n\times 3} \underbrace{\begin{bmatrix} \|v_1\| & q_1\cdot u_2 & q_1 \cdot u_3 \\ 0 & \|v_2\| & q_2 \cdot u_3 \\ 0 & 0 & \|v_3\| \end{bmatrix}}_{3\times 3}$$

$n=3$

{linearly independent and so a basis}

Suppose $\left\{ \overset{u_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \overset{u_2}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \overset{u_3}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \right\}$ $\qquad \|v_1\| = \left\| \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\| = \sqrt{2}$

$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\qquad q_1 = \dfrac{v_1}{\|v_1\|} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

$$v_2 = u_2 - (q_1 \cdot u_2)\, q_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left( \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left( \frac{1}{\sqrt{2}} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$\|v_2\| = \sqrt{\tfrac{1}{4} + \tfrac{1}{4} + 1} = \sqrt{3/2}$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{3/2}} \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ \sqrt{2/3} \end{bmatrix}$$

$v_3 = u_3 - (q_1 \cdot u_3)\, q_1 - (q_2 \cdot u_3)\, q_2$

$\ldots$

$$= \begin{bmatrix}0\\1\\1\end{bmatrix} - \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} - \left(\frac{1}{\sqrt{3/2}}\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}\right)\frac{1}{\sqrt{3/2}}\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix}$$

$$= \begin{bmatrix}0\\1\\1\end{bmatrix} - \left(\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} - \underbrace{\left(\frac{1}{\sqrt{3/2}}\ \frac{1}{2}\right)}_{1/\sqrt{6}}\frac{1}{\sqrt{3/2}}\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix}$$

$$= \begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{2}{3}\,\frac{1}{2}\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\1\end{bmatrix} - \begin{bmatrix}1/6\\-1/6\\1/3\end{bmatrix} = \begin{bmatrix}-2/3\\2/3\\2/3\end{bmatrix}$$

$$\|v_3\| = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{4}{9}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$

$$q_3 = \frac{v_3}{\|v_3\|} = \frac{\sqrt{3}}{2}\begin{bmatrix}-2/3\\2/3\\2/3\end{bmatrix} = \begin{bmatrix}-1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}$$

$$Q = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3}\\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 0 & \sqrt{2/3} & 1/\sqrt{3}\end{bmatrix} \qquad R = \begin{bmatrix}\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2}\\ 0 & \sqrt{3/2} & 1/\sqrt{6}\\ 0 & 0 & 2/\sqrt{3}\end{bmatrix}$$

Check this

$$\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3}\\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 0 & \sqrt{2/3} & 1/\sqrt{3}\end{bmatrix}\begin{bmatrix}\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2}\\ 0 & \sqrt{3/2} & 1/\sqrt{6}\\ 0 & 0 & 2/\sqrt{3}\end{bmatrix}$$

This can be seen as a way of factoring a matrix $A \in \mathbb{R}^{m\times n}$ with linearly independent columns as

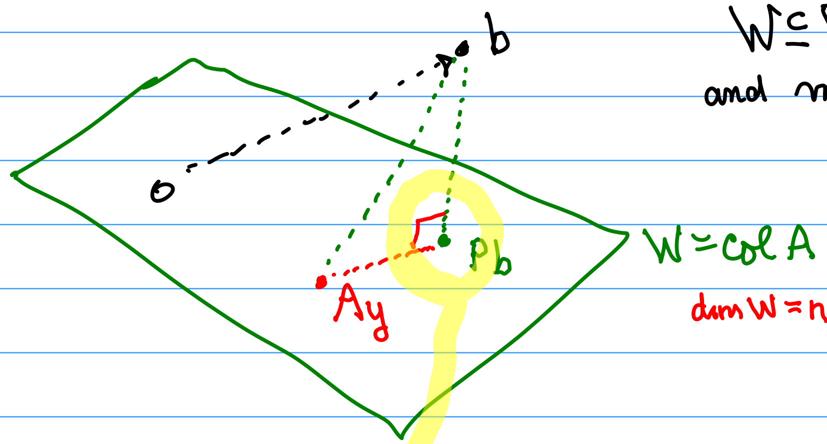
$$\underset{m\times n}{A} = \underset{m\times n}{Q}\ \underset{n\times n}{R}$$

$R$ ← upper triangular (algebraic)

$Q$ ← has orthonormal columns $Q^TQ = I$ (geometric)

$A \in \mathbb{R}^{m \times n}$ with linearly independent columns.

Note if $b \notin \text{Col } A$ then $Ax = b$ has no solutions




$W \subseteq \mathbb{R}^m$ and $m > n$

Idea since we can't solve $Ax = b$ instead let's minimize $\|Ax - b\|$.

$$P = A(A^T A)^{-1} A^T$$

$$(Pb - Ay) \cdot (b - Pb)$$

Since $P$ is a projection onto $W$ then $Pb \in W$

Since $W = \text{col } A$ then $Ay \in W$

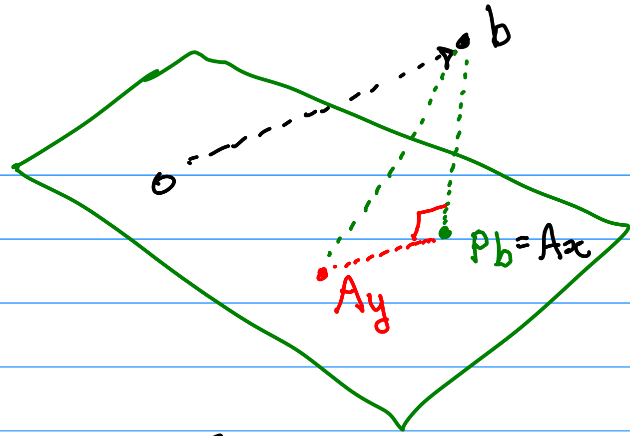
Consequently since $W$ is a subspace $Pb - Ay \in W$

Since $b - Pb$ is orthogonal to any vector in $W$

then $w \cdot (b - Pb) = 0$ for every $w \in W$

so $(Pb - Ay) \cdot (b - Pb) = 0$ So that's really a right angle.

Pythagorean theorem:




$$\|Ay - Pb\|^2 + \|b - Pb\|^2 = \|b - Ay\|^2$$

Therefore $\quad \|b - Pb\| \leq \|b - Ay\| \quad$ for every $y \in \mathbb{R}^n$.

Idea since we can't solve $Ax = b$ instead let's minimize $\|Ax - b\|$.

To find the minimizer all I need is to solve $Ax = Pb$

$$P = A(A^TA)^{-1}A^T$$

$A \in \mathbb{R}^{m \times n}$

$A^T \in \mathbb{R}^{n \times m}$

Thus

$$Ax = A(A^TA)^{-1}A^Tb$$

$$A^TAx = \cancel{A^TA}\,\cancel{(A^TA)^{-1}}A^Tb$$

Thus

$$\underset{n\times m}{A^T}\,\underset{m \times n}{A}x = \underset{n \times m}{A^T}b$$

normal equations

$x \in \mathbb{R}^n$

$b \in \mathbb{R}^m$

$A^TA \in \mathbb{R}^{n \times n}$ and $A^TA$ is invertible because the columns of $A$ are linearly independent.

In summary solving $A^TAx = A^Tb$ is the same as minimizing $\|Ax - b\|$.

Could solve $A^TAx=A^Tb$ using the augmented matrix

$$\left[A^TA \;\vdots\; A^Tb\right]$$

and Gaussian elimination.

Better to use Gram-Schmidt..

$$\underset{m\times n}{A} = \underset{m\times n}{Q}\;\underset{n\times n}{R}$$

upper triangular (algebraic)

has orthonormal columns $Q^TQ=I$ (geometric)

Plug the factorization in

$$A^T = R^TQ^T$$

note $A$ is not invertible (in general)
$Q$ is not invertible (in general)
but $R$ is square and invertible.

$$A^TAx = A^Tb$$

$$R^TQ^TQRx = R^TQ^Tb$$

$$R^TRx = R^TQ^Tb$$

$$R^T(Rx - Q^Tb) = 0$$

$$Rx - Q^Tb = 0$$

since $R^T$ is also invertible

Finally solving $Rx = Q^Tb$ finds the minimum of $\|Ax-b\|$

easy to solve, since $R$ is upper triangular
we can use backsubstitution to find $x$.

Suppose $b \in \text{col } A$. Then $Ax = b$ has a solution,
That means minimum of $\|Ax - b\|$ is zero and
the minimizer is the solution of $Ax = b$.

Use $A = QR$ to solve $Ax = b$ when $b \in \text{Col } A$
by solving $Rx = Q^T b$ for $x$.

Note rephrasing how to find an exact solution as a minimization problem can be useful in numerical contexts where due to rounding errors there is no solution that exactly satisfies the problem.