Suppose $A \in \mathbb{R}^{m\times n}$ and $A \neq A^T$ then

$$\underset{m\times m}{}\ B = A^T A \in \mathbb{R}^{n\times n} \text{ then } B^T = (A^T A)^T = A^T A^{TT} = A^T A = B$$
$$C = A A^T \in \mathbb{R}^{m\times m} \text{ then } C^T = (AA^T)^T = A^{TT} A^T = AA^T = C$$

(A^T: $m\times m$, A: $m\times n$; A: $m\times n$, A^T: $n\times m$)

Symmetric

Generally choose the smaller matrix

if $n < m$ choose $B \in \mathbb{R}^{n\times n}$
if $m < n$ choose $C \in \mathbb{R}^{m\times m}$ } if equal doesn't matter

let's suppose that $n \leq m$ so A has more rows than columns
also (for convenience) that the columns of A are linearly indep.

We know from the QR decomposition chapter that lin. indep. columns means that $A^T A$ is invertible.

If A has more columns than rows then one could replace A by A^T and work with B instead of C. That is C is really the same a B with A replaced by A^T.

By the Spectral theorem applied to B there is an orthonormal basis of eigenvector

$$Bu_i = \lambda u_i \quad \text{and} \quad u_i \in \mathbb{R}^n \quad \text{and} \quad \lambda_i \in \mathbb{R} \text{ for } i=1,\dots,n.$$

Let $U = [u_1 \vdots u_2 \vdots \dots \vdots u_n] \in \mathbb{R}^{n\times n}$

since orthonormal then $U^T U = I$ ✓ definition of orthogonal matrix

since U is square that implies $U^T = U^{-1}$

Thus U is an orthogonal matrix

eigenvectors of B.

Let $y_i = Au_i$ $\quad A \in \mathbb{R}^{m \times n}$

$$\underset{m\times n}{A}\underset{n\times n}{U} = A\left[u_1 \vdots u_2 \vdots \cdots \vdots u_n\right] = \left[Au_1 \vdots Au_2 \vdots \cdots \vdots Au_n\right] \in \mathbb{R}^{m\times n}$$

$$= \left[y_1 \vdots y_2 \vdots \cdots \vdots y_n\right]$$

Turn vectors y_i into unit vectors

$$v_i = \frac{y_i}{\|y_i\|} = \frac{Au_i}{\|Au_i\|} \quad \text{what's that?} \qquad = \frac{Au_i}{\sigma_i} = \frac{y_i}{\sigma_i}$$

Note

$$0 \leq \|Au_i\|^2 = Au_i \overset{\text{def of transpose}}{\cdot} Au_i = u_i \cdot A^T A u_i = u_i \cdot Bu_i$$

$$= \overset{\text{scalar}}{u_i \cdot \lambda_i u_i} = \lambda_i\, u_i \cdot u_i = \lambda_i \geq 0$$

Thus

$$\|Au_i\| = \sqrt{\lambda_i} = \sigma_i \quad \text{where} \quad \sigma_i = \sqrt{\lambda_i}$$

singular values of A.

So $\lambda_i \geq 0$ but I assumed the columns of A are lin indep. then that means B is invertible. So none of the eigenvalues of B are zero. Thus $\lambda_i \neq 0$ and so $\lambda_i > 0$.

Therefore $\sigma_i = \sqrt{\lambda_i}$ not only are real numbers and not imaginary but $\sigma_i > 0$

$y_i = Au_i$

if this is zero then choose y_i to be any vector that is orthogonal to the other ones....

Since $v_i = \dfrac{y_i}{\sigma_i}$ $\qquad y_i = \sigma_i v_i$

$$\underset{m\times n}{A}\,\underset{n\times n}{U} = A\left[\underbrace{u_1 \vdots u_2 \vdots \cdots \vdots u_n}\right] = \left[Au_1 \vdots Au_2 \vdots \cdots \vdots Au_n\right] \in \mathbb{R}^{m\times n}$$

$$= \left[y_1 \vdots y_2 \vdots \cdots \vdots y_n\right] = \left[\sigma_1 v_1 \vdots \sigma_2 v_2 \vdots \cdots \vdots \sigma_n v_n\right]$$

$$= \underbrace{\left[v_1 \vdots v_2 \vdots \cdots \vdots v_n\right]}\underbrace{\begin{bmatrix}\sigma_1 & & & 0\\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_n\end{bmatrix}} = \tilde{V}\tilde{\Sigma}$$

where $\tilde{V} \in \mathbb{R}^{m\times n}$ reduced V, $\tilde{\Sigma} \in \mathbb{R}^{n\times n}$ reduced Σ.

Thus

$$AU = \tilde{V}\tilde{\Sigma}$$

Claim the vectors in $\tilde{V}$ are orthonormal. obvious because $v_i = \frac{y_i}{\|y_i\|}$

$$v_i \cdot v_j = \frac{y_i \cdot y_j}{\|y_i\|\,\|y_j\|} = \frac{Au_i \cdot Au_j}{\sigma_i \sigma_j} = \frac{u_i \cdot Bu_j}{\sigma_i \sigma_j}$$

$$= \frac{\lambda_j}{\sigma_i \sigma_j} u_i \cdot u_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus

has orthonormal columns

$$AU = \tilde{V}\tilde{\Sigma}$$

diagonal

orthogonal $U^T = U^{-1}$

$$A = AUU^T = \underbrace{\tilde{V}\tilde{\Sigma}U^T}$$

reduced singular value decomposition.

$$\underset{m\times n}{A} = \underset{m\times n}{\tilde{V}}\ \underset{n\times n}{\tilde{\Sigma}}\ \underset{n\times n}{U^T}$$

If $m=n$ then nothing to do. But if $m>n$ then the vectors $\{v_1, \dots, v_n\} \subseteq \mathbb{R}^m$ are not a basis of $\mathbb{R}^m$. Extend this to a basis by adding vectors in $\mathbb{R}^m$.

Standard basis of $\mathbb{R}^m$ is $\{e_1, e_2, \dots, e_m\} \subseteq \mathbb{R}^m$

add the linealy independend standard basis vectors $m-n$ times to obtain a basis

$$\{v_1, \dots, v_n, e_{k_1}, e_{k_2}, e_{k_3}, \dots e_{k_{m-n}}\}$$

Use Gram-Schmidt to turn this basis into an orthonormal basis

$$\{q_1, \dots q_n, q_{n+1}, \dots, q_m\}$$

Since the v_i's were already orthonormal then $v_i = q_i$ for $i=1,\dots n$ and so

$$\{v_1 \dots v_n, q_{n+1}, \dots, q_m\}$$

So now

$$V = \left[v_1 \,\vdots\, \cdots \,\vdots\, v_n \,\vdots\, q_{n+1} \,\vdots\, \cdots \,\vdots\, q_m \right]$$

$$\underset{m\times n}{\tilde{V}}\ \underset{n\times n}{\tilde{\Sigma}} = \left[v_1 \,\vdots\, \cdots \,\vdots\, v_n \,\vdots\, q_{n+1} \,\vdots\, \cdots \,\vdots\, q_m \right] \left[\quad ? \quad \right]$$

$$\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} = \underbrace{\begin{bmatrix} v_1 & \cdots & v_n & q_{n+1} & \cdots & q_m \end{bmatrix}}_{V \in \mathbb{R}^{m\times m}} \underbrace{\begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 \; \ddots & \\ 0 & & \sigma_n \\ \hline & 0 & \end{bmatrix}}_{\Sigma \in \mathbb{R}^{m\times n}}$$

Thus

$$\underset{m\times n}{A}\;\underset{n\times n}{U} = \underset{m\times m}{V}\;\underset{m\times n}{\Sigma}$$

This gives the full singular value decomposition

$$A = V \Sigma U^T$$

$\leftarrow$ orthogonal

orthogonal

diagonal but not square

13. Let $B = A^T A$ where A is given by

$$A = \begin{bmatrix} 2 & -1 \\ -2/5 & 11/5 \end{bmatrix}.$$

ind. columns

Note that B has eigenvalues and eigenvectors given by

$$\lambda_1 = 8, \quad x_1 = \begin{bmatrix} -3 \\ 4 \end{bmatrix} \qquad \text{and} \qquad \lambda_2 = 2, \quad x_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

Find the singular value decomposition $A = V\Sigma U^T$ where Σ is a diagonal matrix and U and V are orthogonal matrices.

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$U = \begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} \frac{x_1}{\|x_1\|} & \frac{x_2}{\|x_2\|} \end{bmatrix} = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

$$\|x_1\| = \left\| \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\| = \sqrt{9+16} = \sqrt{25} = 5 \qquad \text{also} \qquad \|x_2\| = 5$$

$$V = \left[v_1 \,\vdots\, v_2 \right]$$

$$v_i = \frac{y_i}{\|y_i\|} = \frac{Au_i}{\|Au_i\|}$$

we know A and u_i so just compute v_i from that.

$$= \begin{bmatrix} 2 & -1 \\ -2/5 & 11/5 \end{bmatrix} \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -6-4 \\ \frac{1}{5}(6+44) \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -10 \\ 50/5 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -10 \\ 10 \end{bmatrix}$$

$$= \frac{10}{5} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

only need direction to normalize

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$y_2 = \begin{bmatrix} 2 & -1 \\ -2/5 & 11/5 \end{bmatrix} \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus,

$$V = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$