Math 330: Quiz 1 Version A Sample Exam

This is a closed-book closed-notes no-calculator-allowed in-class exam. Efforts have been made to keep the arithmetic simple. If it turns out to be complicated, that's either because I made a mistake or you did. In either case, do the best you can and check your work where possible. While getting the right answer is nice, this is not an arithmetic test. It's more important to clearly explain what you did and what you know.

1. Indicate in writing that you have understood the requirement to work independently by writing "I have worked independently on this quiz" followed by your signature as the answer to this question.

2. Suppose $u, v \in \mathbf{R}^3$ and $A \in \mathbf{R}^{2\times 3}$ are given by

$$u = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \qquad v = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & -2 \end{bmatrix}.$$

(i) Find $2u - v$.

$$2\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 9 \end{bmatrix}$$

(ii) Find Au.

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2-2+1 \\ 4+0-6 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

3. Answer the following true false questions:

(i) An inconsistent system has more than one solution.

(A) True

(B) False

inconsistent means no solutions

optional

(ii) Every elementary row operation is reversible.

(A) True

(B) False

(iii) Whenever a system has free variables, the solution set contains a unique solution.

(A) True

(B) False

Free variables correspond to infinite # of solution if there are any solutions at all.

(iv) When two linear transformations are performed one after another, the combined effect may not always be a linear transformation.

(A) True

(B) False

That [is] how we define matrix multiplication as composition and get another matrix.

4. Suppose the coefficient matrix of a system of linear equations has a pivot position in every row. Explain why the system is consistent.

If there is a pivot in every row then no row is going to be the zero row. Thus you never get an augmented matrix that looks like $\left[\begin{array}{c|c} ** & * \\ 000000 & * \end{array}\right]$ with zero on the left and non-zero on the right.

5. Could a set of three vectors in $\mathbf{R}^4$ span all of $\mathbf{R}^4$? Explain. What about n vectors in $\mathbf{R}^m$ when n is strictly less than m?

By the basis theorem every basis of $\mathbb{R}^4$ must have 4 vectors. Therefore 3 vectors could not be a basis. In particular even if they are linearly independent they would not span the space.

Alternative solution to #5 that doesn't use 2.8 or 2.9.

Put the 3 vectors as columns in the matrix A. Thus $A \in \mathbb{R}^{4\times 3}$. The vectors span $\mathbb{R}^4$ if for every $b \in \mathbb{R}^4$ it is possible to solve $Ax=b$. Since A has more rows than columns, the echelon form of A will not have a pivot in every row. Thus at least one row of the echelon form is the zero rows. This means I can't solve $Ax=b$ for all right hand sides. Thus the vectors don't span.

In the general case $A \in \mathbb{R}^{m\times n}$ where $m > n$. Again the row echelon form has at least one zero row because there are more rows than columns. It follows that the columns of A don't span $\mathbb{R}^m$.

Since there are 3 columns there can be at most 3 pivots. Therefore one row doesn't have a pivot and must be the zero row.

Math 330: Quiz 1 Version A Sample Exam

6. Write down the augmented matrix $\left[\, A \,|\, b \,\right]$ corresponding to the system of linear equations given by

The columns don't line up here...

$$\begin{cases} 2x_1 + 4x_2 + 5x_3 - x_4 = 1 \\ 3x_1 + x_2 + 5x_3 - 4x_4 = 2 \\ -21x_1 + 5x_3 + 14x_4 = -7 \end{cases}$$

but *do not* solve these equations.

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \\ [A|b] = \left[\begin{array}{cccc|c} 2 & 4 & 5 & -1 & 1 \\ 3 & 1 & 5 & -4 & 2 \\ -21 & 0 & 5 & 14 & -7 \end{array}\right] \end{array}$$

7. Suppose $A \in \mathbf{R}^{2\times 3}$ is given by

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -3 & 2 \end{bmatrix}.$$

How many free variables does the equation $Ax = 0$ have? Find all solutions to the equation $Ax = 0$.

$$\begin{bmatrix} 2 & 0 & -1 \\ 4 & -3 & 2 \end{bmatrix} \quad r_2 \leftarrow r_2 - 2r_1$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & -3 & 4 \end{bmatrix} \quad \begin{array}{l} r_1 \leftarrow \frac{1}{2} r_1 \\ r_2 \leftarrow -\frac{1}{3} r_2 \end{array}$$

$$\begin{array}{ccc} P & P & F \end{array}$$
$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -4/3 \end{bmatrix} \quad \begin{array}{l} x_1 - \frac{1}{2}x_3 = 0 \\ x_2 - \frac{4}{3}x_3 = 0 \end{array}$$

Solve for pivots

$$x_1 = \frac{1}{2}x_3$$
$$x_2 = \frac{4}{3}x_3$$

Vector form

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_3 \begin{bmatrix} 1/2 \\ 4/3 \end{bmatrix}$$

where $x_3 \in \mathbb{R}$ is free.

Math 330: Quiz 1 Version A Sample Exam

8. The LU factorization of a matrix A is given by

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

with the (, ,) parenthesis means a column vector

Explain how to use this factorization to solve the equation $Ax = b$ and then find the value of x corresponding to $b = (1, 0, 4)$.

Since $Ax=b$ and $A=LU$ then $LUx=b$. Now set $y=Ux$ and obtain $Ly=b$. Therefore, to find x solve the system of systems $\begin{cases} Ly=b \\ Ux=y \end{cases}$.

Solve $Ly=b$ first by substitution

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

$y_1 = 1$
$-3y_1 + y_2 = 0$
$4y_1 - y_2 + y_3 = 4$

$y_1 = 1$
$y_2 = 3y_1 = 3$
$y_3 = 4 + y_2 - 4y_1 = 4+3-4 = 3$

Thus $y = (1,3,3)$.

Next solve $Ux = y$ by back substitution

$$\begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

$x_3 = 3$
$-3x_2 + 4x_3 = 3$
$2x_1 - x_2 + 2x_3 = 1$

$x_3 = 3$

$x_2 = \dfrac{3 - 4x_3}{-3} = \dfrac{-9}{-3} = 3$

$x_1 = \dfrac{1 + x_2 - 2x_3}{2} = \dfrac{1+3-6}{2} = \dfrac{-2}{2} = -1$

Solution $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$.