Solutions to Homework #1

- Section 1.4 # 14

14. Let $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ and $A = \begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}$. Is $\mathbf{u}$ in the subset of $\mathbb{R}^3$ spanned by the columns of A? Why or why not?

To check if u is in span of the columns of A we try to solve $Ax = u$. If possible, the answer is yes; if not, the answer is no.

$$[A|u] = \begin{bmatrix} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 1 & 3 & 0 & 2 \end{bmatrix}$$

$r_1 \leftrightarrow r_3$

note this row operation is only for convenience of hand calculations

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 5 & 8 & 7 & 2 \end{bmatrix}$$

$r_3 \leftarrow r_3 - 5r_1$

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & -7 & 7 & -8 \end{bmatrix}$$

$r_3 \leftarrow r_3 + 7r_2$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -29 \end{array}\right]$$

Since there is a row of zeros on the left in the last row but the value -29 on the right, then $Ax = u$ has no solution. Thus, u is <u>not</u> in the span of columns of A.

- Section 1.4 # 44

44. Suppose A is a 3×3 matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^3$ with the property that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Explain why the columns of A must span $\mathbb{R}^3$.

If $Ax=b$ has a unique solution then there are no free variables. Since A has 3 columns this means there are 3 pivot variables. Since each pivot is in a different row, than 3 rows have pivot variables in them. That means all rows have pivot variables. Consequently its possible to solve $Ax=b$ for any b. Thus, the columns of A span all of $\mathbb{R}^3$.

- Section 1.4 # 46

46. Let A be a 5×3 matrix, let $\mathbf{y}$ be a vector in $\mathbb{R}^3$, and let $\mathbf{z}$ be a vector in $\mathbb{R}^5$. Suppose $A\mathbf{y} = \mathbf{z}$. What fact allows you to conclude that the system $A\mathbf{x} = 4\mathbf{z}$ is consistent?

If $Ay = z$, then setting $x = 4y$ yields

$$Ax = A4y = 4Ay = 4z$$

where the second equality is due to linearity.

In particular, writing $f(x) = Ax$ yields

$$A4y = f(4y) = 4f(y) = 4Ay$$

by the second linearity property

$$f(\alpha x) = \alpha f(x) \quad \text{for all } \alpha \in \mathbb{R} \text{ and } x \in \mathbb{R}^n.$$

Thus, it's the linearity of matrix-vector multiplication that allows one to conclude $Ax = 4z$.