Section 3.2 # 37

37. Show that if A is invertible, then $\det A^{-1} = \dfrac{1}{\det A}$.

Since $AA^{-1} = I$, then $\det AA^{-1} = \det I$.

Now $\det I = \underbrace{1 \cdot 1 \cdot 1 \cdots 1}_{n \text{ times}} = 1$

and $\det AA^{-1} = \det A \det A^{-1}$.

Therefore $\det A \det A^{-1} = 1$.

It follows that

$$\det A^{-1} = \frac{1}{\det A}.$$

Section 3.2 # 41

41. Let U be a square matrix such that $U^T U = I$. Show that $\det U = \pm 1$.

Taking determinant of both sides gives

$$\det U^T U = \det I = 1.$$

Since $\det U^T = \det U$ then

$$\det U^T U = \det U^T \det U = (\det U)^2$$

Therefore

$$(\det U)^2 = 1$$

it follows taking square roots of both sides that

$$\det U = \pm\sqrt{1} = \pm 1.$$

Section 3.2 # 49

49. Verify that $\det A = \det B + \det C$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & u_1 + v_1 \\ a_{21} & a_{22} & u_2 + v_2 \\ a_{31} & a_{32} & u_3 + v_3 \end{bmatrix},$$

$$B = \begin{bmatrix} a_{11} & a_{12} & u_1 \\ a_{21} & a_{22} & u_2 \\ a_{31} & a_{32} & u_3 \end{bmatrix}, \quad C = \begin{bmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ a_{31} & a_{32} & v_3 \end{bmatrix}$$

Note, however, that A is *not* the same as $B + C$.

By definition

$$\det A = \sum_{i=1}^{3} (-1)^{i+3} a_{i3} \det A_{i3}$$

Plugging in the fact that $a_{i3} = u_i + v_i$ yields

$$\det A = \sum_{i=1}^{3} (-1)^{i+3} (u_i + v_i) \det A_{i3}$$

$$= \sum_{i=1}^{3} (-1)^{i+3} u_i \det A_{i3} + \sum_{i=1}^{3} (-1)^{i+3} v_i \det A_{i3}$$

Now since crossing out the 3rd row of either A, B or C results in the same matrix, then

$$A_{i3} = B_{i3} = C_{i3} \quad \text{for } i = 1, 2, 3.$$

Consequently

$$\det A = \sum_{i=1}^{3} (-1)^{i+3} u_i \det A_{i3} + \sum_{i=1}^{3} (-1)^{i+3} v_i \det A_{i3}$$

$$= \sum_{i=1}^{3} (-1)^{i+3} b_{i3} \det B_{i3} + \sum_{i=1}^{3} (-1)^{i+3} c_{i3} \det C_{i3}$$

$$= \det B + \det C.$$

Since

$$B + C = \begin{bmatrix} a_{11} & a_{12} & u_1 \\ a_{21} & a_{22} & u_2 \\ a_{31} & a_{32} & u_3 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ a_{31} & a_{32} & v_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2a_{11} & 2a_{12} & u_1 + v_1 \\ 2a_{21} & 2a_{22} & u_2 + v_2 \\ 2a_{31} & 2a_{32} & u_3 + v_3 \end{bmatrix} \neq A$$

we note that A is not the same as $B+C$.