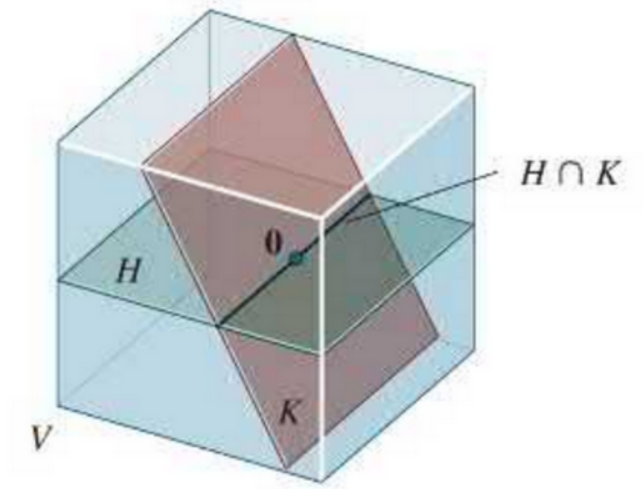
Section 4.1 # 40

40. Let H and K be subspaces of a vector space V. The **intersection** of H and K, written as $H \cap K$, is the set of $\mathbf{v}$ in V that belong to both H and K. Show that $H \cap K$ is a subspace of V. (See the figure.) Give an example in $\mathbb{R}^2$ to show that the union of two subspaces is not, in general, a subspace.



To show $H \cap K$ is a subspace it is sufficient to check three properties

(a) Claim $0 \in H \cap K$. Since H is a subspace then $0 \in H$. Similarly K is a subspace so $0 \in K$. It follows that $0 \in H \cap K$.

(b) Claim $H \cap K$ is closed under vector addition. Suppose $u, v \in H \cap K$. Then $u, v \in H$ and $u, v \in K$. Since $u, v \in H$ and H is a subspace, then $u+v \in H$. Similarly $u+v \in K$. Consequently $u+v \in H \cap K$.

(c) Claim H is closed under multiplication by scalars. Suppose $u \in H \cap K$ and $c \in \mathbb{R}$. Then $u \in H$ implies $cu \in H$. Similarly $u \in K$ implies that $cu \in K$. It follows that $cu \in H \cap K$.

Having checked the properties, it follows $H \cap K$ is a subspace.

Note that $H \cup K$ is not, in general a subspace.

Here is an example in $\mathbb{R}^2$ showing that $H \cup K$ need not be a subspace. Let

$$H = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix} : x_1 \in \mathbb{R} \right\} \quad \text{and} \quad K = \left\{ \begin{bmatrix} 0 \\ x_2 \end{bmatrix} : x_2 \in \mathbb{R} \right\}$$

Note that H and K are subspaces. In fact the are the column spaces

$$H = \text{Col} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad K = \text{col} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

By definition

$$H \cup K = \{ x : \text{either } x \in H \text{ or } x \in K \text{ or both} \}.$$

$$= \left\{ x ; \ x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \text{ for some } x_1 \in \mathbb{R} \text{ or } x = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \text{ for some } x_2 \in \mathbb{R} \right\}.$$

In particular, any vector in $H \cup K$ has at least one element equal to zero. Now consider property (b) and let

$$u \in \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v \in \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since $u \in H$ and $v \in \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $u, v \in H \cup V$. On the other hand $u+v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin H \cup V$ since neither component of of that vector is zero.

Note, there are many other examples of subspaces H and K in $\mathbb{R}^2$ such that $H \cup K$ is not a subspace.

In general taking H and K to be different one-dimensional subspaces of $\mathbb{R}^2$ results in an example where $H \cup K$ is not a subspace.

Section 4.2 # 47

47. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Given a subspace U of V, let $T(U)$ denote the set of all images of the form $T(\mathbf{x})$, where $\mathbf{x}$ is in U. Show that $T(U)$ is a subspace of W.

Since $T(U) \subseteq W$ is a subset of W, what's left is to show the three properties are satisfied.

(a) Claim $0 \in T(U)$. Since U is a subspace of V then $0 \in U$. Then $T(0) = 0$ due to the fact that T is linear. It follows that $0 \in T(U)$.

(b) Claim $T(U)$ is closed under vector addition. Now $u, v \in T(U)$, means there is $x, y \in U$ such that $u = T(x)$ and $v = T(y)$. Since U is a subspace, it follows $x+y \in U$. Consequently, $T(x+y) \in T(U)$. By linearity of T it follows $T(x+y) = T(x) + T(y) = u + v \in T(U)$.

(c) Claim $T(U)$ is closed under multiplication by scalars. Let $u \in T(U)$ and $c \in \mathbb{R}$. Then there exists $x \in U$ such that $T(x) = u$. Since U is a subspace it is closed under multiplication by scalars. Thus $cx \in U$ and so $T(cx) \in T(U)$. By linearity $T(cx) = cT(x) = cu \in T(U)$.

Section 4.2 # 48

48. Given $T : V \to W$ as in Exercise 47, and given a subspace Z of W, let U be the set of all $\mathbf{x}$ in V such that $T(\mathbf{x})$ is in Z. Show that U is a subspace of V.

By definition $U = \{x \in V : T(x) \in Z\}$. To see this is a subspace check the three properties.

(a) Claim $0 \in U$. Since V is a space then $0 \in V$. Moreover, the linearity of T implies $T(0) = 0$. Now, since Z is a subspace, it follows that $T(0) \in Z$. Consequently $0 \in U$.

(b) Claim U is closed under vector addition. Let $x, y \in U$. Then by definition $x, y \in V$ and further that $T(x), T(y) \in Z$. Since V is a space then $x + y \in V$ and since Z is a subspace $T(x) + T(y) \in Z$. By linearity of T we further have $T(x) + T(y) = T(x+y) \in Z$, From this it follows that $x + y \in U$.

(c) Claim U is closed under multiplication by scalars. Let $x \in U$ and $c \in \mathbb{R}$ be a scalar. By definition of U it holds that $x \in V$ and $T(x) \in Z$. Since V is a vector space then $cx \in V$. Similarly, since Z is a subspace we have $cT(x) \in Z$. The linearity of T then implies $cT(x) = T(cx) \in Z$. Consequently $cx \in U$.