Math 430 Midterm Version A

1. Suppose U_1 and U_2 are subspaces of V. The sum $U_1 + U_2$ is

(A) the largest subspace of V contained in $U_1 \cup U_2$.

(B) the smallest subspace of V containing $U_1 \cup U_2$.

(C) the largest subspace of V contained in $U_1 \cap U_2$.

(D) the smallest subspace of V containing $U_1 \cap U_2$.

(E) none of these.

2. A list of vectors $(v_1, \dots, v_m)$ in V are linearly independent if and only if

(A) every choice of $a_1, \dots, a_m \in \mathbf{F}$ makes $a_1v_1 + \cdots + a_mv_m$ equal 0.

(B) the only choice of $a_1, \dots, a_m \in \mathbf{F}$ that makes $a_1v_1 + \cdots + a_mv_m$ equal 0 is $a_1 = \cdots = a_m = 0$.

(C) there exists $a_1, \dots, a_m \in \mathbf{F}$, not all 0, such that $a_1v_1 + \cdots + a_mv_m = 0$.

(D) there exists $a_1, \dots, a_m \in \mathbf{F}$, not all 0, such that $a_1v_1 + \cdots + a_mv_m \neq 0$.

(E) none of these.

3. Let $T \in \mathcal{L}(V, W)$ and $(v_1, \dots, v_n)$ be a basis of V and $(w_1, \dots, w_m)$ be a basis of W. With respect to these basis, the matrix $M(T)$ is defined as the $m \times n$ matrix with entries $a_{j,k}$ where

(A) $w_j = a_{j,1}Tv_1 + \cdots + a_{j,n}Tv_n$.

(B) $w_k = a_{1,k}Tv_1 + \cdots + a_{n,k}Tv_n$.

(C) $Tv_j = a_{j,1}w_1 + \cdots + a_{j,m}w_m$.

(D) $Tv_k = a_{1,k}w_1 + \cdots + a_{m,k}w_m$.

(E) none of these.

4. Two vector spaces are called isomorphic if

(A) there is a third vector space containing both of them.

(B) they contain the same number of elements.

(C) there is an invertible linear map from one vector space onto the other one.

(D) their intersection is the trivial subspace.

(E) none of these.

5. QED at the end of a mathematical proof is

(A) an abreviation for quite easily done.

(B) an abreviation for quantum electrodynamics.

(C) an abreviation for quod erat demonstrandum.

(D) the initials the famous Greek mathematician Q. E. Democritis who invented the deductive method of proof.

(E) none of the these.

6. Consider the matrix A with reduced row echelon form R given by

$$A = \begin{bmatrix} 1 & 3 & 1 & -2 & 4 \\ -1 & -5 & -3 & 5 & 10 \\ 2 & -2 & -6 & 7 & 56 \\ 1 & 7 & 5 & -9 & -32 \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} 1 & 0 & -2 & 0 & 5 \\ 0 & 1 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(i) Find a basis for the subspace range(A) and state its dimension.

(ii) Find a basis for the subspace null(A) and state its dimension.

7. Answer the following true or false questions. Provide a counterexample if false and a brief justification if true.

(i) True or False: Every operator on a finite-dimensional, nonzero, real vector space has an invarient subspace of dimension 1.

(ii) True or False: Let V be a real vector space with $\dim V = 1$ and let $T \in \mathcal{L}(V)$. Then there exists $\lambda \in \mathbf{R}$ such that $Tv = \lambda v$ for all $v \in V$.

(iii) True or False: Suppose that U and W are subspaces of V. Then $V = U \oplus W$ if and only if $V = U + W$ and $U \cap W = \{0\}$.

8. Answer the following true or false questions. Provide a counterexample if false and a brief justification if true.

(i) True or False: Suppose V and W are finite dimensional vector spaces. If $T \in \mathcal{L}(V, W)$ then T is injective if and only if T is surjective.

(ii) True or False: If V and W are finite dimensional vector spaces, then $\mathcal{L}(V, W)$ is finite dimensional and $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$.

(iii) True or False: Let V be a finite dimensional vector space and $T, S \in \mathcal{L}(V)$. Then $\dim \operatorname{null}(TS) = \dim \operatorname{null}(T) + \dim \operatorname{null}(S)$.

9. Prove the following theorem:

> **Theorem 5.10:** Every operator on a finite-dimensional, nonzero, complex vector space has an eigenvalue.

10. Work one of the following homework problems:

> **§1#9.** Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.
>
> **§5#4.** Suppose that $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\operatorname{null}(T - \lambda I)$ is invariant under S for every $\lambda \in \mathbf{F}$.

11. Work one of the following homework problems:

§**3#21.** Suppose that V is finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

§**3#22.** Suppose that V is finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$.

12. For extra credit prove the following theorem:

> **Theorem 3.4:** If V is finite dimensional and $T \in \mathcal{L}(V, W)$, then range(T) is a finite-dimensional subspace of W and
>
> $$\dim V = \dim \operatorname{null}(T) + \dim \operatorname{range}(T).$$