Least Squares Fitting

1a. Consider the linear model

$$F_1(x) = \sum_{i=0}^{3} c_i x^i$$

with unknown parameters c_i. The file `file-1.dat` consists of the data points (x_j, y_j, σ_j) where y_j equals $F_1(x_j)$ plus a normally distributed error with mean 0 and variance σ_j^2. This file has the following format:

1. The first line consists of a single integer n telling how many data points there are.
2. Each subsequent line specifies x_j, y_j and σ_j by three floating point numbers separated by spaces.

Let

$$\tilde{A} = \begin{bmatrix} 1/\sigma_1 & x_1/\sigma_1 & x_1^2/\sigma_1 & x_1^3/\sigma_1 \\ 1/\sigma_2 & x_2/\sigma_2 & x_2^2/\sigma_2 & x_2^3/\sigma_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1/\sigma_n & x_n/\sigma_n & x_n^2/\sigma_n & x_n^3/\sigma_n \end{bmatrix} \quad \text{and} \quad \tilde{y} = \begin{bmatrix} y_1/\sigma_1 \\ y_2/\sigma_2 \\ \vdots \\ y_n/\sigma_n \end{bmatrix}.$$

Use the method of least squares to minimize

$$\chi^2 = \|\tilde{A}c - \tilde{y}\|_2^2 = \sum_{j=1}^{n} \left(\frac{y_j - F_1(x_j)}{\sigma_j} \right)^2$$

and find the parameters c_i that maximize the likelihood of the data. Estimate the errors $\sigma(c_i)$ in the parameters.

1b. Repeat part a using the linear model

$$F_2(x) = \sum_{i=0}^{3} c_i \cos\left(x\sqrt{i}\right).$$

and the data from the file `file-2.dat`.

1c. Use $F_1(x)$ to model the data in `file-2.dat` and $F_2(x)$ to model the data in `file-1.dat`. Find the parameters that minimize χ^2 and plot the data and the model for each of the above cases. What can be said about the goodness of fit?

1d. [Extra Credit and for CS/Math 666] Let n be the number of data points and m be the number of parameters. The probability that the minimum of χ^2 is less than C^2 is given by

$$P\{\chi^2 < C^2\} = \frac{1}{\Gamma(a)} \int_0^{C^2/2} e^{-t} t^{a-1} dt$$

where $a = (n-m)/2$. Let C_{jk}^2 be the minimum of χ^2 found when using F_j to model the data in `file-`k`.dat`. Compute $P\{\chi^2 < C_{jk}^2\}$ for $j, k = 1, 2$. This provides a quantitative measure of the goodness of fit.