Krylov Subspace Methods

1a. Consider the matrix defined by

$$a_{ij} = \begin{cases} 4 & \text{if } i = j \\ -1 & \text{if } i = j+1 \text{ or } i+1 = j \\ 0 & \text{otherwise.} \end{cases}$$

Let $n = 1000$ and write a Matlab program to find Ax for any vector $x \in \mathbf{R}^n$. Alternatively, read about sparse matrices in Matlab and a figure how to represent A using a sparse matrix.

1b. Let b be the vector in $\mathbf{R}^n$ defined by $b_i = 1/i^2$ and define

$$\mathcal{K}_m = \operatorname{span}\{b, Ab, A^2b, \ldots, A^{m-1}b\}.$$

Let $x_m \in \mathcal{K}_m$ be a vector which minimizes $\|Ax - b\|$ among all possible vectors in $\mathcal{K}_m$. Compute $\|Ax_m - b\|$ for $m = 1, 2, \ldots, 10$.

1c. Let x^* be the best approximation to $Ax = b$ found in step 1b. Define the residual $r = b - Ax^*$ and let y^* be the point in

$$\operatorname{span}\{r, Ar, A^2r, \ldots, A^9r\}$$

which minimizes $\|Ay - r\|$. Find the norm of the error $\|A(x^* + y^*) - b\|$ for the iteratively improved solution $x^* + y^*$ to $Ax = b$.

1d. Let $n = 100$ and repeat step 1b for the matrix A and vector b given by `A=randn(100)` and `b=rand(100,1)`. Does the Krylov subspace method of minimum residuals work to solve $Ax = b$ in this case? Use the Matlab command `plot(eig(A),'+')` to plot the eigenvalues of A.

1e. Using the matrix A from part d define $B = A + 20I$. Plot the eigenvalues of B. How are they related to the eigenvalues of A? Does the Krylov subspace method work to solve $Bz = b$? What is the relation between z and the solution x of $Ax = b$? Can you find x from z without inverting A?

1f. [Extra Credit and for CS/Math 666] Repeat the steps 1a–c for the 5000×5000 matrix defined by

$$a_{ij} = \begin{cases} 8 & \text{if } i = j \\ -1 & \text{if } i = j+1 \text{ or } i+1 = j \\ -1/2 & i = 2j+1 \text{ or } 2i+1 = j \\ -1/4 & i = 3j+2 \text{ or } 3i+2 = j \\ 0 & \text{otherwise.} \end{cases}$$

Print using `format long` the 1-st component and the 1000-th component of the vectors x^* and y^* for reference.