Some Calculus Theorems

We have spent some time understanding the errors involved when approximating real numbers on a computer. We shall also need to approximate functions as well. In 1885 Weierstrass proved that any continuous function on an interval $[a, b]$ could be approximated arbitrarily well by a polynomial of sufficiently high degree. In particular, given any continuous function $f : [a, b] \to \mathbf{R}$ and $\epsilon > 0$, then there exists a polynomial p such that

$$|p(x) - f(x)| < \epsilon \qquad \text{for all} \qquad x \in [a, b].$$

This theorem motivates the use of polynomials to represent unknown functions in the algorithms we shall be developing for the computer.

Let us state four important Calculus theorems that we shall use to understand errors related to approximating functions in our algorithms.

Intermediate Value Theorem. Let $f : [a, b] \to \mathbf{R}$ be continuous. If $f(a)$ and $f(b)$ have opposite signs, then there exists $c \in (a, b)$ such that $f(c) = 0$.

Mean Value Theorem 1. Let f be continuous on $[a, b]$ and continuously differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f(b) - f(a) = (b - a)f'(c)$.

Mean Value Theorem 2. Let f and g be continuous on $[a, b]$ and continuously differentiable on (a, b). Then there exists $c \in (a, b)$ such that $(f(b)-f(a))g'(c) = (g(b)-g(a))f'(c)$.

Taylor's Theorem. Let f be continuous on $[a, b]$ and $n + 1$ times continuously differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$\begin{aligned} f(b) = f(a) + (b - a)f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots \\ + \frac{(b-a)^n}{n!} f^{(n)}(a) + \frac{(b-a)^{(n+1)}}{(n+1)!} f^{(n+1)}(c). \end{aligned}$$

We have already used the intermediate value theorem to provide estimates of the errors in the bisection method. We now discuss other methods for solving $f(x) = 0$.

Let me tell you a story which might not be true: Once upon a time I was taking an introductory algebra class and the teacher asked me to solve the quadratic equation

$$x^2 + 100x + 1 = 0.$$

I tried to solve it as

$$100x = -1 - x^2 \qquad \text{and therefore} \qquad x = -\frac{1 + x^2}{100}.$$

The teacher, who was trying to teach factoring and the quadratic formula, marked my solution wrong.

In introductory algebra, that is the end of the story, but in numerical methods the transformation of the original problem $f(x) = 0$ into the form $x = \Phi(x)$ can be used as an iterative method for approximating x. Namely, let x_0 be some initial guess and define

$$x_1 = \Phi(x_0), \qquad x_2 = \Phi(x_1), \qquad \ldots \qquad x_{n+1} = \Phi(x_n).$$

If x_n converges as $n \to \infty$ then we obtain

$$\alpha = \Phi(\alpha) \qquad \text{where} \qquad \alpha = \lim_{n\to\infty} x_n.$$

Therefore $x = \alpha$ is a solution for $f(x) = 0$. This is demonstrated by

Example 6a

```
>> format long
>> phi=inline('-(1+x*x)/100');
>> x=0.0;
>> x=phi(x)
x = -0.0100000000000000
>> x=phi(x)
x = -0.0100010000000000
>> x=phi(x)
x = -0.0100010002000100
>> x=phi(x)
x = -0.0100010002000500
```

One is not always so lucky as in Example 6a. The iteration might not converge. However, if it does, then we have found a solution. We now use the first mean value theorem to find conditions under which the iterative scheme $x_{n+1} = \Phi(x_n)$ converges.

Let α be the solution such that $\Phi(\alpha) = \alpha$ and let x_0 be an initial guess of α. Provided that Φ is differentiable, then the mean value theorem implies for each approximation x_n that there exists c_n between α and x_n such that $\Phi(x_n) - \Phi(\alpha) = (x_n - \alpha)\Phi'(c_n)$. Therefore

$$\begin{aligned}
x_1 - \alpha &= \Phi(x_0) - \Phi(\alpha) = (x_0 - \alpha)\Phi'(c_0) \\
x_2 - \alpha &= \Phi(x_1) - \Phi(\alpha) = (x_1 - \alpha)\Phi'(c_1) \\
&\vdots \\
x_{n+1} - \alpha &= \Phi(x_n) - \Phi(\alpha) = (x_n - \alpha)\Phi'(c_n)
\end{aligned}$$

Substituting the first line into the second, the second into the third, and so forth, we obtain

$$\begin{aligned}
x_1 - \alpha &= (x_0 - \alpha)\Phi'(c_0) \\
x_2 - \alpha &= (x_0 - \alpha)\Phi'(c_0)\Phi'(c_1) \\
&\vdots \\
x_{n+1} - \alpha &= (x_0 - \alpha)\Phi'(c_0)\Phi'(c_1)\cdots\Phi'(c_n).
\end{aligned}$$

Therefore, if $|\Phi'(x)| \le \lambda < 1$ for all x such that $|x - \alpha| \le |x_0 - \alpha|$, then

$$|x_{n+1} - \alpha| \le |x_0 - \alpha|\lambda^n \to 0 \qquad \text{as} \qquad n \to \infty.$$

Note that this argument not only proves that x_n converges, but provides an estimate on the rate of convergence.