Newton's Method

1a. Consider the function $f\colon \mathbf{R}^2 \to \mathbf{R}$ defined by

$$f(x,y) = \frac{1+3x-y}{3+x^2+y^2}.$$

Write subroutines to compute f,

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \\ \frac{\partial f}{\partial y} \end{bmatrix} \qquad \text{and} \qquad J = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}.$$

You may use the Maple `codegen` subroutine library to help write your code. Test your program for (x,y) equal to $(1,2)$, $(-1,4)$ and $(3,-1)$.

b. Write a subroutine to solve the system of linear equations $Ax = b$ where A is a 2×2 matrix. Test your program on the problems

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Did you use some form of pivoting? If not, why not?

c. The function $f(x,y)$ has a maximum near $(1,-1)$. Use Newton's method

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \big[J(x_n,y_n)\big]^{-1} \nabla f(x_n,y_n)$$

to solve $\nabla f = 0$ and find this maximum. Instead of computing J^{-1} in the above equation use part b to solve

$$J(x_n,y_n) \begin{bmatrix} a \\ b \end{bmatrix} = \nabla f(x_n,y_n)$$

and then set

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix}.$$

at each step. Why is it better not to compute J^{-1}?

d. [Extra credit and for Math/CS 666] Use Newton's method to find the absolute maximum for

$$f(x,y) = \frac{1+\ln(x^2+7y^2+1)}{1+3x^2+2y^2-xy+x}.$$

Note that this function has two relative maxima. Don't choose the wrong one as the global maximum. Consider starting with a number of different initial guesses and keeping the one that leads to the greatest relative maximum.