## Rounding Examples

Given $x \in \mathbf{R}$ we denote by $x^{*}$ the nearest floating point approximation to $x$. In the case of a tie, $x^{*}$ is chosen to be the approximation in which the least significant digit in the mantissa is even. In base 10, this is called the banker's round. When working with base 2 this rule ensures that the least significant bit of $x^{*}$ is zero in case of a tie.

The IEEE 754 standard actually describes 4 types of rounding. For default, it specifies the rounding method we have denoted by $x^{*}$. This method has the advantage that rounding errors tend to cancel out. In particular, a sum of such rounding errors can be viewed as a random walk with equal chances for adding either positive or negative errors.

There are certain cases where rounding $x \in \mathbf{R}$ to the nearest floating point representation $x^{*}$ is not desirable. For example, suppose we wanted an upper bound on $x$. Then computing an upper bound on $x^{*}$ might not yield an upper bound on $x$ because it could happen that $x^{*}<x$. To address these issues, IEEE 754 specifies three other types of rounding: round to $\infty$, round to $-\infty$ and round to 0 .

Let $\mathbf{F}$ be the set of all floating point numbers. Then round to $\infty$, round to $-\infty$ and round to 0 may be defined as

$$
\begin{aligned}
\mathrm{r}_{\infty}(x) & =\min \{f \in \mathbf{F}: x \leq f\} \\
\mathrm{r}_{-\infty}(x) & =\max \{f \in \mathbf{F}: f \leq x\}, \\
\mathrm{r}_{0}(x) & =\operatorname{signum}(x) \cdot \mathrm{r}_{-\infty}(|x|),
\end{aligned}
$$

respectively. Round to 0 is also called truncating. Note that for $x \in \mathbf{R}$ we have

$$
\mathrm{r}_{-\infty}(x) \leq x \leq \mathrm{r}_{\infty}(x)
$$

and similarly

$$
\mathrm{r}_{-\infty}(x) \leq x^{*} \leq \mathrm{r}_{\infty}(x)
$$

Consider the floating point calculation divide 1 by 3 to find an approximation of $1 / 3$. Using base 10 floating point with 6 significant digits we have

$$
3.33333 \times 10^{-1}=\mathrm{r}_{-\infty}(1 / 3) \leq 1 / 3 \leq \mathrm{r}_{\infty}(1 / 3)=3.33334 \times 10^{-1}
$$

Thus, the true value of $1 / 3$ lies in the interval $\left[3.33333 \times 10^{-1}, 3.33334 \times 10^{-1}\right]$. Similar techniques can be used to provide rigorous mathematical bounds for any floating point calculation. Google on intlab and interval arithmetic for more information.

All computing hardware that conforms to the IEEE 754 standard supports the default rounding method we have denoted by $x^{*}$ plus these three additional rounding methods; however, most general purpose programming languages such as C and Fortran don't provide a built-in way to access the additional rounding modes. Matlab doesn't either. Fortunately, the default method of rounding $x$ to the nearest representation $x^{*}$ is preferred for the numerical algorithms we will be developing in this course.

Let us estimate the errors for an addition problem using three-digit decimal normalized floating point arithmetic with the default rounding.

$$
6.19 \times 10^{2}+{ }^{*} 5.82 \times 10^{2}=\left(6.19 \times 10^{2}+5.82 \times 10^{2}\right)^{*}=\left(12.01 \times 10^{2}\right)^{*}=1.20 \times 10^{3}
$$

The generated error in this calculation is

$$
\begin{aligned}
e_{\mathrm{gen}} & =\left(6.19 \times 10^{2}+^{*} 5.82 \times 10^{2}\right)-\left(6.19 \times 10^{2}+5.82 \times 10^{2}\right) \\
& =-0.001 \times 10^{3}=-1 \times 10^{0}
\end{aligned}
$$

To estimate the accumulated (or total) error we need to know what the initial and propagated errors are. If $x^{*}=6.19 \times 10^{2}$ then we may infer that $x \in\left(6.185 \times 10^{2}, 6.195 \times 10^{2}\right)$. Note that by the banker's round that $\left(6.185 \times 10^{2}\right)^{*}=6.18 \times 10^{2}$ and $\left(6.195 \times 10^{2}\right)^{*}=$ $6.20 \times 10^{2}$. Thus we don't include the endpoints in this interval. Similarly, if $y^{*}=5.82 \times 10^{2}$ then we may infer that $y \in\left[5.815 \times 10^{2}, 5.825 \times 10^{2}\right]$. In this case we include the endpoints in the interval. It follows that the initial errors are bounded as

$$
\left|e_{x}\right|<0.005 \times 10^{2}=5 \times 10^{-1} \quad \text { and } \quad\left|e_{y}\right| \leq 5 \times 10^{-1}
$$

Hence, the propagated error satisfies

$$
\left|e_{\text {prop }}\right|=\left|e_{x}+e_{y}\right|<1 \times 10^{0} .
$$

We conclude that the cumulative error $e_{\text {tot }}=e_{\text {gen }}+e_{\text {prop }}$ satisfies

$$
-2 \times 10^{0}<e_{\mathrm{tot}}<0
$$

Note that this guarantees, no matter what the initial values of $x$ and $y$ were, that the cumulative error is negative.

For a second example, let us work a multiplication problem.

$$
\left(3.60 \times 10^{3}\right) *\left(1.01 \times 10^{-1}\right)=\left(3.6360 \times 10^{2}\right)^{*}=3.64 \times 10^{2} .
$$

Hence the relative generated error satisfies

$$
0.00110011001 \leq \tilde{e}_{\text {gen }}=\frac{0.004}{3.6360} \leq 0.00110011002
$$

Note that we have rounded the decimal approximation of the upper bound up towards $\infty$ and the lower bounds down towards $-\infty$ to ensure that the inequality is mathematically correct. We shall continue with this kind of estimates throughout this example. Since $x^{*}=3.60 \times 10^{3}$ we infer that $x \in\left[3.595 \times 10^{3}, 3.605 \times 10^{3}\right]$. Therefore

$$
-0.00139082059 \leq \frac{-0.005}{3.595} \leq \tilde{e}_{x} \leq \frac{0.005}{3.605} \leq 0.00138696256
$$

Similarly $y^{*}=1.01 \times 10^{-1}$ implies $y \in\left(1.005 \times 10^{-1}, 1.015 \times 10^{-1}\right)$. Therefore

$$
-0.00497512438 \leq \frac{-0.005}{1.005}<\tilde{e}_{y}<\frac{0.005}{1.015} \leq 0.00492610838
$$

Hence, the relative propagated error satisfies

$$
-0.00635902547<\tilde{e}_{\text {prop }}=\tilde{e}_{x} \tilde{e}_{y}+\tilde{e}_{x}+\tilde{e}_{y}<0.00631990327
$$

We conclude that the relative cumulative error $\tilde{e}_{\text {tot }}=\tilde{e}_{\text {gen }} \tilde{e}_{\text {prop }}+\tilde{e}_{\text {gen }}+\tilde{e}_{\text {prop }}$ satisfies

$$
-0.00526591109<\tilde{e}_{\text {tot }}<0.00742696588
$$

Notice that simpler estimates on the errors can be made by neglecting the terms $\tilde{e}_{x} \tilde{e}_{y}$ and $\tilde{e}_{\text {gen }} \tilde{e}_{\text {prop }}$ as suggested in the textbook. Of course, in this case, the final estimates on the errors are only approximate and don't represent true mathematical bounds.

