Taylor Series and Newton's Method

Your work should be presented in the form of a typed report using clear and properly punctuated English. Where appropriate include full program listings and output. If you choose to work in a group of two, please turn in independently prepared reports.

1a. For $t \in [-1, 1)$ Taylor's theorem implies

$$\log(1-t) = -\Big(\sum_{j=1}^{n} \frac{t^j}{j}\Big) - \frac{1}{1-c}\Big(\frac{t^{n+1}}{n+1}\Big)$$

where c is an unknown constant between 0 and t. Let $t = 0.5$. Solve for n to ensure that the Taylor polynomial of degree n approximates $\log(0.5)$ with a relative error less than 5×10^{-16}.

1b. [Extra Credit and Math/CS 666] Show that

$$2^{n+1}(n+1) \geq 10^{16}$$

implies the relative error of the Taylor polynomial satisfies

$$|\mathrm{Rel}(T_n(x))| = \left|\frac{\log x - T_n(x)}{\log x}\right| \leq 5 \times 10^{-16}$$

for all $x \in [0.5, 1.5]$.

1c. Write a program that uses a suitable Taylor polynomial $T_n(x)$ to approximate $\log x$ for $x \in [0.5, 1.5]$. Make your computation as accurate as possible. Compare your results to the builtin log function and compute the relative error for $x = 0.5$, $x = 1.03$ and $x = 1.4$.

1d. Use Newton's method to find $\sqrt{2}$ by solving $x^2 - 2 = 0$. Then use the identity $\log 2 = 2 \log \sqrt{2}$ to compute $\log 2$ as accurately as possible.

1e. For $x > 0$ let k be the unique integer such that $2^{k-1} < x \leq 2^k$. Let $w = x/2^k$ so that $0.5 < w \leq 1$. Use the identity $\log x = k \log 2 + \log w$ and parts 1c and 1d to create a program that computes $\log x$ for all values of $x > 0$. Compare your results to the builtin log function and compute the relative error for $x = 17$, $x = 1083$ and $x = 0.19$.