Math/CS 466/666 Homework 1

Vectors, Matrices and Condition Number

1. Consider the vectors and matrices

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & -1 & 3 \\ 4 & 0 & 2 \\ -1 & 2 & 0 \end{bmatrix}$$

(i) Find $\|u\|_1$ and $\|A\|_1$ and verify that $\|Au\|_1 \le \|A\|_1 \|u\|_1$.

(ii) Find $\|u\|_\infty$ and $\|A\|_\infty$ and verify that $\|Au\|_\infty \le \|A\|_\infty \|u\|_\infty$.

(iii) Compute $A^\dagger A$.

(iv) Prove the eigenvalues of $A^\dagger A$ are real and non-negative.

(v) Use the power method to find the largest eigenvalue λ_1 such that $A^\dagger A x = \lambda_1 x$ for some non-zero eigenvector x.

(vi) Find $\|u\|_2$ and compute $\|A\|_2$ using the formula $\|A\|_2 = \sqrt{\lambda_1}$.

(vii) Verify that $\|Au\|_2 \le \|A\|_2 \|u\|_2$.

Math/CS 466 students solve either part (viii) or (ix) and treat the other part as extra credit; Math/CS 666 students solve both parts.

(viii) Use the inverse power method to find the smallest eigenvalue λ_3 of $A^\dagger A$.

(ix) Prove or disprove that the condition number

$$\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2 = \sqrt{\lambda_1/\lambda_3}.$$