Do not use the computer in any way for this part of the quiz. Also keep your book and notes closed.

1. State Taylor's theorem with remainder for a scalar function in one variable.

2. State the RK4 tableau for the classical fourth-order Runge–Kutta method.

3. Suppose A is an 4×4 matrix and $\sigma(B) = \{1, 2, 3, 4\}$ are the eigenvalues of the matrix $B = A^\dagger A$ where $A^\dagger$ denotes the complex transpose of A. Find $\|A\|_2$.

4. Consider the 3/8-rule RK tableau given by

$$\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{3} & \frac{1}{3} & & & \\
\frac{2}{3} & -\frac{1}{3} & 1 & & \\
1 & 1 & -1 & 1 & \\
\hline
 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8}
\end{array}.$$

Translate this tableau into an algebraic or pseudocode form suitable for implementation on a digital computer.

5. Consider the $n \times n$ tridiagonal matrix A and the vectors x and y given by

$$A = \begin{bmatrix} a_0 & c_0 & 0 & \cdots & 0 \\ b_0 & a_1 & c_1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_{n-3} & a_{n-2} & c_{n-2} \\ 0 & \cdots & 0 & b_{n-2} & a_{n-1} \end{bmatrix}, \quad x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{bmatrix}.$$

State the Thomas algorithm for solving $Ax = y$.

6. Let $p_k(x)$ for $k = 0, 1, 2, \ldots$ be a family of orthogonal polynomials of such that

$$p_k(x) \text{ has degree } k \qquad \text{and} \qquad \int_a^b p_i(x)p_j(x)\,dx = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases}$$

Fix n and let x_i for $i = 0, 1, \ldots, n$ be the roots of $p_{n+1}(x)$. Define

$$w_i = \int_a^b \ell_i(x)\,dx \qquad \text{where} \qquad \ell_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}.$$

Show that the Gaussian quadrature approximation

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} w_i f(x_i)$$

is exact when $f(x)$ is a polynomial of degree $2n + 1$ or less.

Please use the Ubuntu VM for this part of the quiz. You may also use your notes and textbooks as well as online resources such as Wikipedia and Google. However, do not use email or any other messaging service.

Submit your program and output using the commands

```
/nfs/home/ejolson/opt/bin/submit -q1 program.c
/nfs/home/ejolson/opt/bin/submit -q2 output.txt
```

Here `program.c` is the name of your program and `output.txt` is an output file obtained by running the program with the command

```
./a.out >output.txt
```

If you wish to change any part of your submission simply retype the appropriate submit command again. You may check each of your submissions with the command

```
/nfs/home/ejolson/opt/bin/submit -pn
```

where `n` is equal the number used in submit command.

7. Please answer one of the following questions.

(i) Write or modify a C computer program to use the finite difference method to approximate the solution to the two-point boundary value problem

$$y'' + \frac{2}{x}y' - \frac{2}{x^2}y = \frac{\sin(\log x)}{x^2}$$

on the interval $[1, 2]$ where $y(1) = 1$ and $y(2) = 2$. Use a grid spacing of size $h = 1/m$ where $m = 10$.

(ii) Write or modify a C computer program to use the classical RK4 method to approximate $y(4)$ where $y(t)$ is the solution to the initial value problem

$$y' = y - y\sin(ty)$$

on the interval $[1, 4]$ with initial condition $y(1) = 1$. Use time steps of size $h = 3/n$ where $n = 100$.

8. [Math/CS 666 and Extra Credit] Please solve the other problem appearing above and submit your answer using the commands

```
/nfs/home/ejolson/opt/bin/submit -q3 program2.c
/nfs/home/ejolson/opt/bin/submit -q4 output2.txt
```