1. Let $A \in \mathbf{R}^{d\times d}$ and $x \in \mathbf{R}^d$. Define the vector and corresponding matrix norms as

$$\|x\| = \sum_{i=1}^{d} x_i^2 \qquad \text{and} \qquad \|A\| = \max\{\, \|Ax\| : \|x\| = 1 \,\}.$$

Prove that $\|A\| = \rho(B)^{1/2}$ where $B = A^t A$ and $\rho(B) = \max\{\, |\lambda| : \det(B-\lambda I) = 0 \,\}$.

Since B is real and symmetric, the spectral theorem implies there is an orthonormal basis ξ_i of $\mathbf{R}^d$ consisting of eigenvectors x_i with eigenvalues λ_i such that

$$B\xi_i = \lambda_i \xi_i \qquad \text{for} \qquad i = 1,\ldots,d \qquad \text{and} \qquad \xi_i\cdot\xi_j = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases}$$

Let $x \in \mathbf{R}^d$. Since the ξ_i form a basis, then there exists $c_i \in \mathbf{R}$ such that

$$x = \sum_{i=1}^{d} c_i \xi_i.$$

Since the ξ_i are orthonormal, then

$$\|x\|^2 = x\cdot x = \Big(\sum_{i=1}^{d} c_i\xi_i\Big)\cdot\Big(\sum_{j=1}^{d} c_j\xi_j\Big) = \sum_{i=1}^{d}\sum_{j=1}^{d} c_i c_j \xi_i\cdot\xi_j = \sum_{i=1}^{d} c_i^2.$$

Moreover,

$$\|Ax\|^2 = Ax\cdot Ax = A^tAx\cdot x = Bx\cdot x = B\Big(\sum_{i=1}^{d} c_i\xi_i\Big)\cdot\Big(\sum_{j=1}^{d} c_j\xi_j\Big)$$
$$= \Big(\sum_{i=1}^{d} c_iB\xi_i\Big)\cdot\Big(\sum_{j=1}^{d} c_j\xi_j\Big) = \Big(\sum_{i=1}^{d} c_i\lambda_i\xi_i\Big)\cdot\Big(\sum_{j=1}^{d} c_j\xi_j\Big) = \sum_{i=1}^{d}\lambda_i c_i^2.$$

Since $\|Ax\|^2 \geq 0$ and the c_i's are arbitrary, then the above equality implies $\lambda_i \geq 0$. Thus, we may assume that the λ_i's have been ordered such that

$$0 \leq \lambda_1 \leq \cdots \leq \lambda_d.$$

Now

$$\|A\| = \max\{\, \|Ax\|^2 : \|x\|^2 = 1 \,\}^{1/2} = \max\Big\{\, \sum_{i=1}^{d}\lambda_i c_i^2 : \sum_{i=1}^{d} c_i^2 = 1. \Big\}^{1/2}.$$

This shows that $\|A\|$ is the maximum over all weighted averages of the λ_i's. Since the weighted average is never greater than the maximum, it follows that

$$\|A\| \leq \max\{\, \lambda_i : i = 1,\ldots,d \,\}^{1/2}.$$

Moreover, since the maximum is obtained by choosing $c_i = 0$ for $i < d$ and $c_d = 1$ we obtain

$$\|A\| = \max\{\, \lambda_i : i = 1,\ldots,d \,\}^{1/2} = \max\{\, |\lambda_i| : i = 1,\ldots,d \,\}^{1/2}$$
$$= \max\{\, |\lambda| : \det(B-\lambda I) = 0 \,\}^{1/2} = \rho(B)^{1/2}.$$

2. Let $B \in \mathbf{R}^{d\times d}$ be a positive symmetric matrix.

(i) State the power method for finding the spectral radius $\rho(B)$

Let $y_0 \in \mathbf{R}^d$ be chosen so that it doesn't lie in any strict eigen-subspace of B. Define

$$x_k = y_k/\|y_k\| \qquad \text{and} \qquad y_{k+1} = Bx_k \qquad \text{for} \qquad k = 0, 1, 2, \ldots.$$

Then $y_{k+1} \cdot x_k \to \rho(B)$ as $k \to \infty$.

(ii) Prove that the power method converges to the spectral radius.

Let the eigenvectors ξ_i and eigenvalues λ_i of B be given as in part (i). Define p such that

$$\lambda_i < \lambda_d \quad \text{for} \quad i < p \qquad \text{and} \qquad \lambda_i = \lambda_d \quad \text{for} \quad i \ge p.$$

Choose $y_0 \in \mathbf{R}^d$ at random. With probability 1 it follows that y_0 is not in a strict eigen-subspace of B. Therefore

$$y_0 = \sum_{i=1}^d c_i \xi_i \qquad \text{where} \qquad c_i \ne 0 \quad \text{for} \quad i = 1, \ldots, d.$$

By definition $x_0 = y_0/\|y_0\|$ and consequently

$$y_1 = Bx_0 = B\sum_{i=1}^d c_i\xi_i/\|y_0\| = \sum_{i=1}^d c_i B\xi_i/\|y_0\| = \sum_{i=1}^d c_i\lambda_i\xi_i/\|y_0\|.$$

Therefore

$$x_1 = y_1/\|y_1\| = \sum_{i=1}^d c_i\lambda_i\xi_i \Big/ \Big\| \sum_{i=1}^d c_i\lambda_i\xi_i \Big\|.$$

By induction, it follows that

$$x_k = \sum_{i=1}^d c_i\lambda_i^k\xi_i \Big/ \Big\| \sum_{i=1}^d c_i\lambda_i^k\xi_i \Big\| \qquad \text{and} \qquad y_{k+1} = \sum_{i=1}^d c_i\lambda_i^{k+1}\xi_i \Big/ \Big\| \sum_{i=1}^d c_i\lambda_i^k\xi_i \Big\|.$$

Thus

$$y_{k+1} \cdot x_k = \frac{\sum_{i=1}^d c_i^2\lambda_i^{2k+1}}{\sum_{i=1}^d c_i^2\lambda_i^{2k}} = \frac{\sum_{i=1}^d c_i^2\lambda_i(\lambda_i/\lambda_d)^{2k}}{\sum_{i=1}^d c_i^2(\lambda_i/\lambda_d)^{2k}}.$$

By the definition of p we have that

$$\lim_{k\to\infty} (\lambda_i/\lambda_d)^{2k} = \begin{cases} 0 & \text{for } i < p \\ 1 & \text{for } i \ge p. \end{cases}$$

Consequently

$$y_{k+1} \cdot x_k \to \frac{\sum_{i=p}^d c_i^2\lambda_d}{\sum_{i=p}^d c_i^2} = \lambda_d = \rho(B) \qquad \text{as} \qquad k \to \infty.$$