Math/CS 466/666 Quiz 1 Version A Answer Key

1. State Newton's method for solving $f(x)=0$.

Definition. Let f be a twice continuously differentiable function. Newton's method is given by the fixed point iteration

$$x_{n+1}=g(x_n)\qquad\text{where}\qquad g(x)=x-f(x)/f'(x)$$

and x_0 is an initial approximation of the root.

2. Let x_∞ be a point such that $f(x_\infty)=0$ and $f'(x_\infty)\neq 0$. Prove that Newton's method is quadratically convergent provided x_0 is close enough to x_∞.

Proof. Let $\delta>0$ be chosen small enough such that

$$|g'(x)|=\left|\frac{f(x)f''(x)}{f'(x)^2}\right|\le\gamma<1\qquad\text{for}\qquad|x-x_\infty|\le\delta.$$

Then

$$|x_{n+1}-x_\infty|=|g(x_n)-g(x_\infty)|=\left|\int_{x_n}^{x_\infty}g'(s)ds\right|\le\gamma|x_n-x_\infty|$$

shows $|x_n-x_\infty|\le\gamma^n|x_0-x_\infty|\to 0$ as $n\to\infty$ and moreover that $|x_n-x_\infty|\le\delta$. Now define $e_n=x_n-x_\infty$. By Taylor's theorem there exists ξ_n between x_n and x_∞ such that

$$0=f(x_\infty)=f(x_n)-f'(x_n)e_n+\frac{f''(\xi_n)}{2}e_n^2\qquad\text{for}\qquad n=0,1,2,\ldots.$$

Therefore

$$\frac{f(x_n)}{f'(x_n)}=e_n-\frac{f''(\xi_n)}{2f'(x_n)}e_n^2.$$

It follows that

$$e_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}-x_\infty=\frac{f''(\xi_n)}{2f'(x_n)}e_n^2$$

At this point there are two ways to proceed: one following the proof in the text and the other following the presentation in class. Both are presented here.

Proof continued as in class. Let

$$A=\max\big\{\,|f''(x)|:|x-x_\infty|\le\delta\,\big\}\quad\text{and}\quad B=\min\big\{\,|f'(x)|:|x-x_\infty|\le\delta\,\big\}.$$

Since f'' is continuous then $A<\infty$. By definition of δ we have $f'(x)\neq 0$ for $|x-x_\infty|\le\delta$. Therefore, continuity of f' implies $B>0$. It follows that

$$|e_{n+1}|=\left|\frac{f''(\xi_n)}{2f'(x_n)}e_n^2\right|\le\frac{A}{2B}|e_n|^2\qquad\text{for}\qquad n=0,1,2,\ldots.$$

Consequently $|e_{n+1}|\le M|e_n|^2$ where $M=A/(2B)$. This shows Newton's method is at least quadratically convergent. Alternatively, we may proceed as in the book.

Proof continued as in the book. The limit

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = \lim_{n\to\infty} \left|\frac{f''(\xi_n)}{2f'(x_n)}\right| = \left|\frac{f''(x_\infty)}{2f'(x_\infty)}\right| < \infty$$

shows Newton's method is at least quadratically convergent.

Note that the book actually defines quadratic order of convergence in terms of the above limit. Thus, the proof is officially done at this point. If, however, one would like to exhibit the inequality $|e_{n+1}| \le M|e_n|^2$ used in class one can proceed as follows: By definition of limit there is N large enough such that

$$\left|\frac{f''(\xi_n)}{2f'(x_n)}\right| \le \left|\frac{f''(x_\infty)}{2f'(x_\infty)}\right| + 1 \qquad \text{whenever} \qquad n \ge N.$$

Then it is sufficient to take

$$M = \max\left\{\left|\frac{f''(x_\infty)}{2f'(x_\infty)}\right| + 1\right\} \cup \left\{\left|\frac{f''(\xi_n)}{2f'(x_n)}\right| : n = 0, 1, \ldots, N-1\right\}.$$

3. Explain why it is sometimes said that Newton's method doubles the number of significant digits at each iteration.

Explanation. Let

$$\alpha = \log_{10}\big(5M|x_\infty|\big) \qquad \text{so that} \qquad 10^\alpha = 5M|x_\infty|.$$

Suppose x_n is accurate to k significant digits. By the definition this means

$$\frac{|x_n - x_\infty|}{|x_\infty|} \le 5 \times 10^{-k}.$$

Now

$$\begin{aligned}\frac{|x_{n+1} - x_\infty|}{|x_\infty|} &\le \frac{M|x_n - x_\infty|^2}{|x_\infty|} = M|x_\infty|\Big(\frac{|x_n - x_\infty|}{|x_\infty|}\Big)^2 \\ &\le M|x_\infty|(5^2 \times 10^{-2k}) = 5 \times 10^{\alpha - 2k}\end{aligned}$$

implies x_{n+1} is accurate to $2k-\alpha$ significant digits. Provided k is large compared to α this is about twice the number of significant digits that were accurate in x_n. Since $k \to \infty$ as $x_n \to x_\infty$, it is natural to assume that k is very large compared to α. Therefore Newton's method about doubles the number of significant digits between each iteration.

Math/CS 466/666 Quiz 1 Version A Answer Key

Consider the following table of divided differences

i	x_i	$f[x_i]$				
0	1.0	3.0				
			1.0			
1	2.0	4.0		-0.75		
			-2.0		$\boxed{}$	
2	5.0	-2.0		0.50		-0.04018
			1.0		-0.14286	
3	8.0	1.0		-0.50		
			-1.0			
4	9.0	0.0				

4. Determine what value goes in the box.

Answer. Calculate the divided difference

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_0, x_1, x_2] - f[x_1, x_2, x_3]}{x_0 - x_1} = \frac{-0.75 - 0.50}{1 - 8} = \frac{1.25}{7} \approx 0.17857.$$

5. Use the information in the table to write down the interpolating polynomial of degree 2 that passes through the points $(2, 4)$, $(5, -2)$ and $(8, 1)$.

Answer. Reading the coefficients from the second diagonal in the chart yields

$$p(x) = 4.0 + (-2.0)(x - 2) + (0.50)(x - 2)(x - 5).$$

6. State the method of Gauss quadrature in two steps:

(i) Define the orthogonal polynomials P_n of degree n on the interval $[-1, 1]$.

Definition. Consider the inner product and norm on the space of integrable functions defined by

$$(f, g) = \int_{-1}^{1} f(x)g(x)dx \qquad \text{and} \qquad \|f\| = \sqrt{(f, f)}.$$

The orthogonal polynomials

$$\{ P_k : k = 0, 1, \ldots n \}$$

may be obtained using the Gram–Schmidt orthogonalization procedure with respect to the above inner product and norm starting with the standard polynomial basis

$$\{ x^k : k = 0, 1, \ldots n \}.$$

In particular, define

$$
\begin{aligned}
v_0(x) &= 1 & P_0 &= \frac{v_0}{\|v_0\|}\\
v_1(x) &= x - (x, P_0)P_0 & P_1 &= \frac{v_1}{\|v_1\|}\\
v_2(x) &= x^2 - (x^2, P_0)P_0 - (x^2, P_1)P_1 & P_2 &= \frac{v_2}{\|v_2\|}\\
&\vdots & &\vdots\\
v_n(x) &= x^n - \sum_{k=0}^{n-1}(x^n, P_k)P_k & P_n &= \frac{v_n}{\|v_n\|}
\end{aligned}
$$

(ii) Define the points x_k and the weights w_k for used in the approximation

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{k=0}^{n} w_k f(x_k).$$

Definition. Let x_k for $k = 0, 1, \ldots, n$ be the $n+1$ distinct roots to the orthogonal polynomial P_{n+1} of degree $n+1$. Thus $P_{n+1}(x_k) = 0$ for $k = 0, 1, \ldots, n$. Consider the system of $n+1$ linear equations given by

$$\int_{-1}^{1} x^j\,dx = \sum_{k=0}^{n} w_k x_k^j \qquad \text{for} \qquad j = 0, 1, \ldots, n$$

in the $n+1$ unknowns w_k where $k = 0, 1, \ldots n$. Since the x_j's are distinct this system is non-singular. Therefore, there exists a unique solution for the w_k's. This specifies the x_k's and w_k's that appear in the Gauss quadrature formula.

7. Prove the Gauss quadrature method is exact for polynomials of degree $2n+1$.

Proof. Let p be a polynomial of degree $2n+1$. Since the P_{n+1} has degree $n+1$, the division algorithm implies there exist polynomials r and q of degree n such that

$$p(x) = q(x)P_{n+1}(x) + r(x).$$

Claim that

$$\int_{-1}^{1} r(x)dx = \sum_{k=0}^{n} w_k r(x_k).$$

Write

$$r(x) = \sum_{j=0}^{n} a_j x^j.$$

Then by the choice of w_k's we have

$$\begin{aligned}\int_{-1}^{1} r(x)dx &= \int_{-1}^{1} \sum_{j=0}^{n} a_j x^j\,dx = \sum_{j=0}^{n} a_j \int_{-1}^{1} x^j\,dx \\ &= \sum_{j=0}^{n} a_j \sum_{k=0}^{n} w_k x_k^j = \sum_{k=0}^{n} w_k \sum_{j=0}^{n} a_j x_k^j = \sum_{k=0}^{n} w_k r(x_k).\end{aligned}$$

Since P_{n+1} is orthogonal to all polynomials of degree n or less and $P_{n+1}(x_k) = 0$, then

$$\begin{aligned}\int_{-1}^{1} p(x)dx &= \int_{-1}^{1} \big(q(x)P_{n+1}(x) + r(x)\big)dx = (q, P_{n+1}) + \int_{-1}^{1} r(x)dx \\ &= \int_{-1}^{1} r(x)dx = \sum_{k=0}^{n} w_k r(x_k) = \sum_{k=0}^{n} w_k\big(q(x_k)\cdot 0 + r(x_k)\big) \\ &= \sum_{k=0}^{n} w_k\big(q(x_k)P_{n+1}(x_k) + r(x_k)\big) = \sum_{k=0}^{n} w_k p(x_k).\end{aligned}$$

This finishes the proof.

8. [Extra Credit and for Math/CS 666] Consider the approximation

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$

Explain why taking h too large leads to a bad approximation and why taking h too small also leads to a bad approximation.

Explanation. Since

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x),$$

then analytically the approximation is only guaranteed to be accurate when h is sufficiently small. Therefore, a bad approximation is likely to occur when h is too large.

On the other hand, if h is very small then the continuity of f implies that $f(x+h)$ will be nearly equal to $f(x-h)$. The rounding error that happens when numerically subtracting two numbers which are nearly equal in the numerator causes a loss of precision. Therefore, taking h too small will also lead to a bad approximation.