

### Math/CS 466/666: Homework Solutions for Chapter 3

3.1 Can *all* matrices  $A \in \mathbf{R}^{n \times n}$  be factored  $A = LU$ ? Why or why not?

Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Claim that this matrix can not be factored  $A = LU$ .

For contradiction, suppose that it could. Then there would exist a lower-triangular matrix  $L \in \mathbf{R}^{2 \times 2}$  and an upper-triangular matrix  $U \in \mathbf{R}^{2 \times 2}$  such that  $A = LU$ . Therefore

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} = \begin{bmatrix} L_{11}U_{11} & L_{11}U_{12} \\ L_{21}U_{11} & L_{21}U_{12} + L_{22}U_{22} \end{bmatrix}$$

Now  $L_{11}U_{11} = 0$  implies either  $L_{11} = 0$  or  $U_{11} = 0$  or both. If  $U_{11} = 0$  then  $L_{21}U_{11} = 0$  contradicts the lower left corner of  $A$  being one. On the other hand, if  $L_{11} = 0$  then  $L_{11}U_{12} = 0$  contradicts the upper right corner of  $A$  being one. As both possibilities lead to a contradiction, there is no  $LU$  factorization of  $A$ .

3.6 Numerical algorithms appear in many components of simulation software for quantum physics. The Schrödinger equation and others involve complex numbers in  $\mathbf{C}$ , however, so we must extend the machinery we have developed for solving linear systems of equations to this case. Recall that a complex number  $x \in \mathbf{C}$  can be written as  $x = a + bi$ , where  $a, b \in \mathbf{R}$  and  $i^2 = -1$ . Suppose we wish to solve  $Ax = b$ , but now  $A \in \mathbf{C}^{n \times n}$  and  $x, b \in \mathbf{C}^n$ . Explain how a linear solver that takes only real-valued systems can be used to solve this equation.

First observe that the multiplication of complex numbers

$$(a + bi)(c + di) = ac - bd + i(ad + bc)$$

can be expressed as multiplication of the real matrices

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} ac - bd & ad + bc \\ -(ad + bc) & ac - bd \end{bmatrix}.$$

Following this pattern write  $A = A_1 + A_2i$ , where  $A_1, A_2 \in \mathbf{R}^{n \times n}$  and similarly decompose  $x = x_1 + x_2i$  and  $b = b_1 + b_2i$  where  $x_1, x_2, b_1, b_2 \in \mathbf{R}^n$ . Now consider the real-valued matrix in  $\mathbf{R}^{2n \times 2n}$  and the real-valued vectors in  $\mathbf{R}^{2n}$  given by

$$\mathcal{A} = \begin{bmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{bmatrix}, \quad \mathcal{X} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} b_1 \\ -b_2 \end{bmatrix}.$$

Now, if  $x$  is a solution to  $Ax = b$ , then

$$Ax = (A_1 + A_2i)(x_1 + x_2i) = A_1x_1 - A_2x_2 - i(A_1x_2 + A_2x_1)$$

implies  $A_1x_1 - A_2x_2 = b_1$  and  $A_1x_2 + A_2x_1 = b_2$ . On the other hand, if we solve the system of  $2n$  linear equations given by  $\mathcal{A}\mathcal{X} = \mathcal{B}$ . for  $\mathcal{X}$  we obtain

$$\begin{bmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{bmatrix} \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} = \begin{bmatrix} A_1x_1 - A_2x_2 \\ -A_2x_1 - A_1x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ -b_2 \end{bmatrix}.$$

which again implies  $A_1x_1 - A_2x_2 = b_1$  and  $A_1x_2 + A_2x_1 = b_2$ . Thus, the solution to the real valued system  $\mathcal{A}\mathcal{X} = \mathcal{B}$  may be used to solve the complex system  $Ax = b$ .

3.9 Show that the inverse of an (invertible) lower-triangular matrix is lower triangular.

Let  $M = L^{-1}$  and assume for contradiction that  $M$  is not lower triangular. In this case, there is a column in which the matrix fails to be lower triangular. Denote such a column as  $j$  and take  $i$  to be the least row  $i < j$  such that  $M_{ij} \neq 0$ . Thus  $k < i$  implies  $M_{kj} = 0$ . Moreover, since  $L$  is lower triangular then  $k > i$  implies  $L_{ik} = 0$ . Since  $LM = I$  then

$$0 = [LM]_{ij} = \sum_{k=1}^n L_{ik}M_{kj} = L_{ii}M_{ij}.$$

Now, since  $L$  is invertible and lower triangular it has a non-zero diagonal. It follows that  $L_{ii} \neq 0$  and consequently

$$L_{ii}M_{ij} \neq 0.$$

This contradiction implies that  $M$  must have been lower triangular.

**3.11** Show how the  $LU$  factorization of  $A \in \mathbf{R}^{n \times n}$  can be used to compute the determinant of  $A$ .

As this is the first mention of determinant in the book. To answer this question it is necessary to define the determinant. One standard definition is given by the recursive expansion of the top row and the respective minors. Let  $M_{ij}$  denote the  $(n-1) \times (n-1)$  matrix formed by deleting the  $i$ -th row and  $j$ -column from the matrix  $A$ . Then

$$\det(A) = \begin{cases} \sum_{j=1}^n (-1)^{j-1} A_{1j} \det(M_{1j}) & \text{for } n > 1 \\ A_{11} & \text{for } n = 1. \end{cases}$$

If  $L$  is lower triangular it follows that

$$\det(L) = \begin{cases} L_{11} \det(L_{11}) & \text{for } n > 1 \\ L_{11} & \text{for } n = 1. \end{cases}$$

By induction it immediately follows that  $\det(L)$  is the product of the diagonal entries

$$\det(L) = \prod_{i=1}^n L_{ii}.$$

A similar formula for an upper triangular matrix may be obtained by expanding the determinant along the first column. In the case of the  $A = LU$  we have

$$\det(A) = \det(LU) = \det(L) \det(U) = \prod_{i=1}^n L_{ii} U_{ii}.$$

We note that for the  $LU$  factorization discussed in class, the diagonal entries of  $L$  are each identically 1. In this case  $\det(L) = 1$  and we may compute

$$\det(A) = \prod_{i=1}^n U_{ii}.$$

3.15 Suppose  $A \in \mathbf{R}^{n \times n}$  is invertible and admits a factorization  $A = LU$  with ones along the diagonal of  $L$ . Show that such a decomposition of  $A$  is unique.

Suppose there were two different factorizations

$$A = L_1U_1 \quad \text{and} \quad A = L_2U_2$$

where  $L_i$  are lower triangular with ones on the diagonal and the  $U_i$  are upper triangular. We note that  $L_i$  are invertible since they have ones on the diagonal. It follows that the  $U_i = L_i^{-1}A$  are invertible because  $A$  is invertible and  $U_i^{-1} = A^{-1}L_i$ . Now

$$0 = A - A = L_1U_1 - L_2U_2 = L_1(U_1 - U_2) + (L_1 - L_2)U_2$$

implies

$$(U_2 - U_1)U_2^{-1} = L_1^{-1}(L_1 - L_2).$$

Since  $U_2$  is upper triangular then  $U_2^{-1}$  is also upper triangular. It follows that the product on the left hand side is upper triangular. Similarly, since  $L_1$  is lower triangular then  $L_1^{-1}$  is also lower triangular and the product on the right hand side is lower triangular. Since the left and right sides are equal, then  $L_1^{-1}(L_1 - L_2)$  must be both upper and lower triangular and therefore diagonal. Moreover, since the  $L_i$ 's are ones on the diagonal then  $L_1 - L_2$  is lower triangular with zeros on the diagonal. It follows that the product  $L_1^{-1}(L_1 - L_2)$  also has zeros on the diagonal and is therefore equal zero. Thus  $L_1 = L_2$ . It immediately follows that  $U_1 = U_2$ . Therefore, the decomposition of  $A$  must be unique.