Math/CS 466/666: Lecture 3

In 1669 Isaac Newton devised a technique for approximating the solution of a polynomial equation [2]. In 1685 John Wallis named this method Newton's method and Joseph Raphson simplified it in 1690. In 1740 Thomas Simpson extended the method to general nonlinear equations and systems of equations [3]. In 2000 Dongarra and Sullivan listed Newton's method among the top 10 algorithms of the 20th century [1].

Isaac Newton the mathematician, astronomer, theologian and physicist on the left; on the right John Wallis the clergyman and mathematician.

Newton's Method

Newton's method is given by the fixed point iteration

$$x_{n+1} = g(x_n) \qquad \text{where} \qquad g(x) = x - f(x)/f'(x)$$

and x_0 is an initial approximation of the root.

Convergence of Newton's Method. Let f be a twice continuously differentiable function. Let a be a point such that $f(a) = 0$ and $f'(a) \neq 0$. Prove that Newton's method is quadratically convergent provided x_0 is close enough to a.

Proof. Let $\delta > 0$ be chosen small enough such that

$$|g'(x)| = \left| \frac{f(x)f''(x)}{f'(x)^2} \right| \le \gamma < 1 \qquad \text{for} \qquad |x - a| \le \delta.$$

Then, provided $|x_0 - a| \le \delta$, the inequality

$$|x_{n+1} - a| = |g(x_n) - g(a)| = \left| \int_a^{x_n} g'(s)ds \right| \le \gamma |x_n - a|$$

shows $|x_n - a| \le \gamma^n |x_0 - a| \to 0$ as $n \to \infty$ and moreover that $|x_n - a| \le \delta$. Now define $e_n = x_n - a$. By Taylor's theorem there exists ξ_n between x_n and a such that

$$0 = f(a) = f(x_n) - f'(x_n)e_n + \frac{f''(\xi_n)}{2} e_n^2 \qquad \text{for} \qquad n = 0, 1, 2, \ldots.$$

Therefore

$$\frac{f(x_n)}{f'(x_n)} = e_n - \frac{f''(\xi_n)}{2f'(x_n)} e_n^2.$$

It follows that

$$e_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - a = \frac{f''(\xi_n)}{2f'(x_n)} e_n^2$$

Let

$$A = \max \left\{ \, |f''(x)| : |x-a| \le \delta \, \right\} \quad \text{and} \quad B = \min \left\{ \, |f'(x)| : |x-a| \le \delta \, \right\}.$$

Since f'' is continuous then $A < \infty$. By definition of δ we have $f'(x) \neq 0$ for $|x - a| \le \delta$. Therefore, continuity of f' implies $B > 0$. It follows that

$$|e_{n+1}| = \left| \frac{f''(\xi_n)}{2f'(x_n)} e_n^2 \right| \le \frac{A}{2B} |e_n|^2 \qquad \text{for} \qquad n = 0, 1, 2, \ldots.$$

Consequently $|e_{n+1}| \le M|e_n|^2$ where $M = A/(2B)$. This shows Newton's method is at least quadratically convergent. ////

It is sometimes said that Newton's method doubles the number of significant digits at each iteration. This can be explained as follows: Let

$$\alpha = \log_{10} \big(5M|a| \big) \qquad \text{so that} \qquad 10^\alpha = 5M|a|.$$

Suppose x_n is accurate to k significant digits. By the definition this means

$$\frac{|x_n - a|}{|a|} \le 5 \times 10^{-k}.$$

Now

$$\begin{aligned} \frac{|x_{n+1} - a|}{|a|} &\le \frac{M|x_n - a|^2}{|a|} = M|a| \Big(\frac{|x_n - a|}{|a|} \Big)^2 \\ &\le M|a|(5^2 \times 10^{-2k}) = 5 \times 10^{\alpha - 2k} \end{aligned}$$

implies x_{n+1} is accurate to $2k - \alpha$ significant digits. Provided k is large compared to α this is about twice the number of significant digits that were accurate in x_n. Since $k \to \infty$ as $x_n \to a$, it is natural to assume that k is very large compared to α. Therefore Newton's method about doubles the number of significant digits between each iteration.

References

1. Jack Dongarra and Francis Sullivan, Top Ten Algorithms of the Century, *Computing in Science and Engineering*, 2000.
2. Isaac Newton, De analysi per aequationes numero terminorum infinitas, 1669.
3. Thomas Simpson, Essays on Several Curious and Useful Subjects in Speculative and Mix'd Mathematicks, 1740.