Math/CS 466/666: Programming Project 2

> Your work should be presented in the form of a typed report using clear and properly punctuated English. Where appropriate include full program listings and output. If you choose to work in a group of two, please turn in independently prepared reports.

The `matnorm2` subroutine developed in class can be modified such that it iterates until about 15 digits of accuracy have been found in the approximation for $\|A\|_2$. An example of such a modified subroutine follows:

```
double matnorm2(int n,double A[n][n]){
    double B[n][n],y[n],yk[n];
    bzero(B,sizeof(double)*n*n);
    for(int k=0;k<n;k++){                  // B = A^T A
        for(int i=0;i<n;i++){
            for(int j=0;j<n;j++){
                B[i][j]+=A[k][i]*A[k][j];
            }
        }
    }
    for(int i=0;i<n;i++){                  // Choose x ∈ R^n randomly
        y[i]=2.0*random()/RAND_MAX+1.0;    // and store x in y for now
    }
    double q=0,qk;
    for(int k=1;k<100*n;k++){
        multAx(n,n,B,y,yk);                // y_k = B^k x/||B^{k-1}x||_2
        qk=vecnorm2(n,yk);
        for(int j=0;j<n;j++){
            y[j]=yk[j]/qk;                 // Overwrite y by y_k/||y_k||_2
        }
        if(fabs(qk-q)<5e-15*qk){           // Converge to 15 digits where
            return sqrt(qk);               // ||A||_2 ≈ (||B^k x||_2/||B^{k-1}x||_2)^{1/2}
        }
        q=qk;
    }
    fprintf(stderr,"matnorm2: Failed to converge!\n");
    return sqrt(qk);
}
```

The goal of this project is to modify the above code to create an `invmatnorm2` subroutine which approximates $\|A^{-1}\|_2$ and to then combine these two routines to create a third routine `matcond2` which approximates the condition number of A.

1. Let $A \in \mathbf{R}^{n\times n}$ and $C = AA^T$. If A is nonsingular does it follow that C must also be nonsingular? If so, explain why; if not, give a counter example.

2. Show that $C^{-1} = (A^{-1})^T A^{-1}$ and conclude that $\|A^{-1}\|_2 = \sqrt{\rho(C^{-1})}$.

3. Consider the matrix $B = A^T A$. Do B^{-1} and C^{-1} have the same eigenvalues? If so, explain why; if not, give a counter example.

4. Modify the code in `matnorm2` by replacing `multAx` with `plusolve` to create a routine `invmatnorm2` which approximates $\|A^{-1}\|_2$. You will also need to add a call to `plufact` somewhere before the main loop. Test your routine using the matrix

$$A = \begin{bmatrix} 7 & 1 & 7 \\ 5 & 7 & 7 \\ 3 & 7 & 8 \end{bmatrix} \qquad \text{to show that} \qquad \|A^{-1}\|_2 \approx 0.73677.$$

Please include full program source code and output with your report.

5. Write a routine `matcond2` that computes the condition number

$$\text{cond}_2(A) = \|A^{-1}\|_2 \|A\|_2.$$

Approximate the condition number of the matrix given in the previous problem. Please include source code and output with your report.

6. Find $\|A\|_2$, $\|A^{-1}\|_2$ and $\text{cond}_2(A)$ for the special matrix associated with your netid available for download from the course website.

7. [Extra Credit] For $n = 10, 20, 40, 80, \ldots, 1280$ let K_n be the average condition number with respect to the 2-norm of 100 random $n \times n$ matrices with entries uniformly distributed between -1 and 1. Plot K_n versus n and try to deduce a relationship between the size of the random matrices and the average condition number.