Math/CS 466/666: Programming Project 1

1. The first computer program ever written was by Ada Lovelace who wrote a program for the Analytical Engine to compute Bernoulli numbers. The Bernoulli number B_n is given by $B_n = \mathcal{B}_n(0)$ where $\mathcal{B}_n(x)$ is the unique polynomial of degree n such that

$$\int_x^{x+1} \mathcal{B}_n(t)dt = x^n.$$

Find B_n for $n = 0$, 1, 2 and 3 by substituting a polynomial of degree n into the integral and solving for the coefficients so that equality holds. You may use Maple or some other computer algebra system or do the calculation by hand.

Writing $\mathcal{B}_0(x) = \alpha$ for some constant α and substituting into the equation yields

$$\int_x^{x+1} \alpha dt = \alpha = 1$$

Consequently $\alpha = 1$ implies $B_0 = \mathcal{B}_0(0) = 1$.

Writing $\mathcal{B}_1(x) = \alpha + \beta x$ for constants α and β yields

$$\int_x^{x+1} (\alpha + \beta t)dt = \alpha + \frac{\beta}{2}\big((x+1)^2 - x^2\big) = \alpha + \frac{\beta}{2}(2x+1)$$
$$= \left(\alpha + \frac{\beta}{2}\right) + \beta x = x.$$

Consequently

$$\alpha + \frac{\beta}{2} = 0 \qquad \text{and} \qquad \beta = 1.$$

Therefore $\alpha = -1/2$ and $\mathcal{B}_1(x) = -1/2 + x$ imply $B_1 = \mathcal{B}_1(0) = -1/2$.

Writing $\mathcal{B}_2(x) = \alpha + \beta x + \gamma x^2$ for constants α, β and γ yields

$$\int_x^{x+1} (\alpha + \beta t + \gamma t^2)dt = \left(\alpha + \frac{\beta}{2}\right) + \beta x + \frac{\gamma}{3}\big((x+1)^3 - x^3\big)$$
$$= \left(\alpha + \frac{\beta}{2}\right) + \beta x + \frac{\gamma}{3}(3x^2 + 3x + 1)$$
$$= \left(\alpha + \frac{\beta}{2} + \frac{\gamma}{3}\right) + (\beta + \gamma)x + \gamma x^2 = x^2.$$

Consequently

$$\alpha + \frac{\beta}{2} + \frac{\gamma}{3} = 0, \qquad \beta + \gamma = 0 \qquad \text{and} \qquad \gamma = 1.$$

Therefore $\beta = -1$, $\alpha = 1/6$ and $\mathcal{B}_2(x) = 1/6 - x + x^2$ implies $B_2 = \mathcal{B}_2(0) = 1/6$.

Writing $\mathcal{B}_3(x) = \alpha + \beta x + \gamma x^2 + \delta x^3$ for constants α, β, γ and δ yields

$$\begin{aligned}
&\int_x^{x+1} (\alpha + \beta t + \gamma t^2 + \delta t^3) dt \\
&\qquad = \left(\alpha + \frac{\beta}{2} + \frac{\gamma}{3}\right) + (\beta + \gamma)x + \gamma x^2 + \frac{\delta}{4}\big((x+1)^4 - x^4\big) \\
&\qquad = \left(\alpha + \frac{\beta}{2} + \frac{\gamma}{3}\right) + (\beta + \gamma)x + \gamma x^2 + \frac{\delta}{4}(4x^3 + 6x^2 + 4x + 1\big) \\
&\qquad = \left(\alpha + \frac{\beta}{2} + \frac{\gamma}{3} + \frac{\delta}{4}\right) + (\beta + \gamma + \delta)x + \left(\gamma + \frac{3}{2}\delta\right)x^2 + \delta x^3 = x^3.
\end{aligned}$$

Consequently

$$\alpha + \frac{\beta}{2} + \frac{\gamma}{3} + \frac{\delta}{4} = 0, \quad \beta + \gamma + \delta = 0, \quad \gamma + \frac{3}{2}\delta = 0 \quad \text{and} \quad \delta = 1.$$

Therefore

$$\gamma = -\frac{3}{2}, \quad \beta = \frac{1}{2}, \quad \alpha = -\frac{1}{4} + \frac{1}{2} - \frac{1}{4} = 0$$

and $\mathcal{B}_3(x) = (1/2)x - (3/2)x^2 + x^3$ implies $B_3 = \mathcal{B}_3(0) = 0$.

2. By the Fundamental Theorem of Calculus it follows that

$$\frac{d}{dx}\int_x^{x+1} \mathcal{B}_n(t)dt = \mathcal{B}_n(x+1) - \mathcal{B}_n(x) = \int_x^{x+1} \mathcal{B}'_n(t)dt.$$

Use this fact to show that $\mathcal{B}'_n(x) = n\mathcal{B}_{n-1}(x)$.

By definition

$$\int_x^{x+1} \mathcal{B}_n(t)dt = x^n$$

Therefore

$$\frac{d}{dx}\int_x^{x+1} \mathcal{B}_n(t)dt = nx^{n-1}.$$

Since, as noted above

$$\int_x^{x+1} \mathcal{B}'_n(t)dt = \frac{d}{dx}\int_x^{x+1} \mathcal{B}_n(t)dt$$

it follows upon dividing by n that

$$\int_x^{x+1} n^{-1}\mathcal{B}'_n(t)dt = x^{n-1}.$$

Since $\mathcal{B}_n(t)$ is a polynomial of degree n, then $n^{-1}\mathcal{B}'_n(t)$ is a polynomial of degree $n-1$. Moreover, by definition $\mathcal{B}_{n-1}(x)$ is the unique polynomial of degree $n-1$ such that

$$\int_x^{x+1} \mathcal{B}_{n-1}(t)dt = x^{n-1}.$$

Therefore, we conclude $n^{-1}\mathcal{B}'_n(x) = \mathcal{B}_{n-1}(x)$ or, in otherwords, that $\mathcal{B}'_n(x) = n\mathcal{B}_{n-1}(x)$.

3. By the Fundamental Theorem of Calculus we also have

$$\int_0^x \mathcal{B}_n'(t)dt = \mathcal{B}_n(x) - \mathcal{B}_n(0) \quad \text{or equivalently} \quad \mathcal{B}_n(x) = B_n + \int_0^x n\mathcal{B}_{n-1}(t)dt.$$

Integrate the above equality in x from 0 to 1, then interchange the order of integration to obtain the relation that

$$B_n = \int_0^1 tn\mathcal{B}_{n-1}(t)dt \qquad \text{for} \qquad n > 1.$$

Integrating each side of the equality yields

$$\int_0^1 \mathcal{B}_n(x)dx = \int_0^1 B_n dx + \int_0^1 \int_0^x n\mathcal{B}_{n-1}(t)dt\,dx. \tag{3.1}$$

Setting $x = 0$ in the defining equality

$$\int_x^{x+1} \mathcal{B}_n(t)dt = x^n$$

shows that the left side of (3.1) is exactly equal to zero. Since B_n is a constant, the first term of the right is B_n. It remains to switch the order of integration in the last term. Doing so obtains

$$\begin{aligned}\int_0^1 \int_0^x n\mathcal{B}_{n-1}(t)dt\,dx &= \int_0^1 \int_t^1 n\mathcal{B}_{n-1}(t)dx\,dt \\ &= \int_0^1 (1-t)n\mathcal{B}_{n-1}(t)dt = \int_0^1 n\mathcal{B}_{n-1}(t)dt - \int_0^1 tn\mathcal{B}_{n-1}(t)dt\end{aligned}$$

Now, if $n > 1$ we may set $x = 0$ in the defining equality

$$\int_x^{x+1} \mathcal{B}_{n-1}(t) = x^{n-1}$$

to obtain that

$$\int_0^1 n\mathcal{B}_{n-1}(t)dt = 0.$$

Note if $n = 1$ it is not possible to set $x = 0$ above because that would result in the indeterminate form 0^0. This is why the recurrence is only good for $n > 1$. It follows that

$$B_n = \int_0^1 tn\mathcal{B}_{n-1}(t)dt \qquad \text{for} \qquad n > 1.$$

4. Write $\mathcal{B}_{n-1}(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1}x^{n-1}$ and use the identity

$$\mathcal{B}_n(x) = \int_0^1 tn\mathcal{B}_{n-1}(t)dt + \int_0^x n\mathcal{B}_{n-1}(t)dt$$

derived in the previous step to find formulas for B_n and $\mathcal{B}_n(x)$ in terms of the α_k.

Integrating yields

$$\int_0^1 tn\mathcal{B}_{n-1}(t)dt = \int_0^1 \sum_{k=0}^{n-1} n\alpha_k t^{k+1}\,dt = \sum_{k=0}^{n-1} \frac{n\alpha_k}{k+2} t^{k+2}\Big|_0^1 = \sum_{k=0}^{n-1} \frac{n\alpha_k}{k+2}$$

and

$$\int_0^x n\mathcal{B}_{n-1}(t)dt = \int_0^x \sum_{k=0}^{n-1} n\alpha_k t^k\,dt = \sum_{k=0}^{n-1} \frac{n\alpha_k}{k+1} t^{k+1}\Big|_0^x = \sum_{k=0}^{n-1} \frac{n\alpha_k}{k+1} x^{k+1}.$$

Therefore

$$B_n = \sum_{k=0}^{n-1} \frac{n\alpha_k}{k+2} \qquad \text{and} \qquad \mathcal{B}_n(x) = B_n + \sum_{k=0}^{n-1} \frac{n\alpha_k}{k+1} x^{k+1}.$$

5. Starting with $\mathcal{B}_1(x) = x - 1/2$, write a program that computes the Bernoulli numbers by means of the formulas derived in the previous step. Use your program to print a table listing the values of n and B_n for $n = 1, 2, \ldots, 10$.

The program is

```c
#include <stdio.h>
#include <math.h>

#define N 10
double alpha[N+1]={-0.5,1}; // B1(x)=x-1/2;

int main(){
    printf("#%6s %24s\n","n","Bn");
    for(int n=1;;){
        printf("%7d %24.14g\n",n,alpha[0]);
        n++;
        if(n>N) break;
        double b=0;
        for(int k=0;k<n;k++) b+=alpha[k]/(k+2);
        for(int k=n;k>0;k--) alpha[k]=n*alpha[k-1]/k;
        alpha[0]=n*b;
    }
    return 0;
}
```

and the output is

```
#      n                       Bn
       1                     -0.5
       2         0.16666666666667
       3                        0
       4       -0.033333333333333
       5     -6.9388939039072e-16
       6         0.02380952380952
       7      -2.603472992746e-14
       8       -0.033333333333543
       9     -1.8882395647069e-12
      10        0.075757575738698
```