

Math 466/666: Homework Assignment 2

This homework uses the vector and matrix 2-norms to explore some properties of matrices, eigenvalues and eigenvectors. Note that you will need to use Julia or similar software for one of these problems.

Students are encouraged to work together and consult resources outside of the required textbook for this assignment. Please cite any sources you consulted, including Wikipedia, other books, online discussion groups as well as personal communications. Be prepared to independently answer questions concerning the material on quizzes and exams.

Unless a disability makes it difficult, present all pencil-and-paper work in your own hand writing. To do this scan handwritten pages using a cell phone, document camera or flatbed scanner. Alternatively, you may write on a digital tablet with a writing stylus. If a computer was used to solve any part of a problem, include the code, input and output. Please upload your work as a single pdf file to WebCampus.

Given a matrix $A \in \mathbf{R}^{m \times n}$ recall that the matrix 2-norm is defined as

$$\|A\|_2 = \max \{ \|Ax\|_2 : \|x\|_2 = 1 \}$$

where the vector 2-norms $\|Ax\|_2$ and $\|x\|_2$ are given by

$$\|Ax\|_2 = \sqrt{(Ax)^T Ax} \quad \text{and} \quad \|x\|_2 = \sqrt{x^T x} \quad \text{for} \quad x \in \mathbf{R}^n.$$

For convenience of notation we will, henceforth, drop the subscripts and write

$$\|A\| = \|A\|_2, \quad \|Ax\| = \|Ax\|_2 \quad \text{and} \quad \|x\| = \|x\|_2.$$

1. Show that $\|I\| \geq 1$ where I is the identity operator.
2. Let $A \in \mathbf{R}^{n \times n}$ be invertible. Show that $\|A\| \|A^{-1}\| \geq 1$.
3. Define

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{bmatrix}.$$

Use the definition of the matrix 2-norm to compute $\|A\|$ and $\|A^T\|$.

4. Let $A \in \mathbf{R}^{7 \times 4}$ be a matrix with entries chosen randomly (and independently) from the uniform distribution on the interval $[-1, 1]$. Use any numerical means you prefer to compute $\|A\|$ and $\|A^T\|$ for three different samples of the random matrices just described. For example, in Julia you could enter the commands

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using LinearAlgebra
A=2*rand(7,4).-1;
opnorm(A)
opnorm(A')

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three times. What relationship between $\|A\|$ and $\|A^T\|$ do you notice? Please include all program listings and computer output as part of your answer for this question.

5. Given $A \in \mathbf{R}^{m \times n}$ let $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$ be unit vectors such that

$$\|Ax\| = \|A\| \quad \text{and} \quad \|A^T y\| = \|A^T\|.$$

Thus x and y have been chosen to be vectors for which the maximum in the definition of the matrix 2-norm is attained.

- (i) Use the Cauchy–Schwarz inequality to prove that

$$\|Ax\|^2 \leq \|A^T Ax\| \quad \text{and} \quad \|A^T y\|^2 \leq \|AA^T y\|.$$

- (ii) By repeated applications of the definition of the norm it follows that

$$\|AA^T y\| \leq \|A\| \|A^T\| \quad \text{and} \quad \|A^T Ax\| \leq \|A^T\| \|A\|.$$

Explain why $\|AA^T y\| = \|A^T Ax\|$ and finally why $\|A\| = \|A^T\|$.

6. Let $A \in \mathbf{R}^{m \times n}$ set $B = A^T A$ and define

$$\lambda = \max \{ x^T B x : \|x\| = 1 \}.$$

- (i) Explain why $\lambda \geq 0$ and show that $\|A\| = \|A^T\| = \sqrt{\lambda}$.
- (ii) Choose $\xi \in \mathbf{R}^n$ to be a unit vector such that $\xi^T B \xi = \lambda$ and show that $\|B\xi\| \leq \lambda$.
- (iii) Expand the inner product $(B\xi - \lambda\xi)^T (B\xi - \lambda\xi)$ and show that $\|B\xi - \lambda\xi\| = 0$.
- (iv) Is it true or false that ξ must be an eigenvector of B with λ as an eigenvalue? If true explain why; if false provide a counter example.
7. [Extra Credit and for Math 666] Let $B \in \mathbf{R}^{n \times n}$ be any symmetric matrix with $B^T = B$ and choose $\xi \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$ such that $\lambda = \max \{ x^T B x : \|x\| = 1 \}$ and $\xi^T B \xi = \lambda$.
- (i) Find an example of a matrix B such that $\lambda < 0$.
- (ii) Show for any $v \in \mathbf{R}^n$ that $v^T B v \leq \lambda v^T v$.
- (iii) Set $v = \xi + \epsilon w$ where $w \in \mathbf{R}^n$ and $\epsilon > 0$ and simplify to prove that

$$2w^T (B\xi - \lambda\xi) \leq \epsilon(\lambda w^T w - w^T B w).$$

- (iv) Now set $w = B\xi - \lambda\xi$. Is it true or false that ξ must be an eigenvector of B with λ as an eigenvalue? If true explain why; if false provide a counter example.