Examples where I track $\left\{\begin{array}{l}\text{initial}\\ \text{propagated}\\ \text{generated}\\ \text{accumulated}\end{array}\right\}$ errors.

Example: $5.18 + 6.29 - 1.31$

work with absolute error because abs error behaves well w.r.t. addition

want $x+y$ but only have x^* and y^* on the computer

what's the error in $x^* + y^*$ (propagated by +)

$e_{abs}(x) = |x - x^*|$ $\qquad$ $e_{abs}(y) = |y - y^*|$

exact value $\quad$ approximation of x $\qquad$ exact value $\quad$ approximation of y

We have this bound:

$$e_{abs}(x+y) = |x+y - (x^* + y^*)|$$

$$\leq |x - x^*| + |y - y^*| = e_{abs}(x) + e_{abs}(y)$$

Back to example $\qquad$ evaluate left to right

$$(5.18 + 6.29) - 1.31$$

$e_{abs}(5.18) \leq 0.005$ $\qquad$ since by assumption 5.18 was correctly rounded

$e_{abs}(6.29) \leq 0.005$ $\qquad$ since by assumption 6.29 was correctly rounded

Propagation of absolute error in addition

$$e_{abs}(5.18+6.29) \leq e_{abs}(5.18) + e_{abs}(6.29) \leq 0.005 + 0.005 = 0.01$$

$$e_{abs}(1.31) \leq 0.005 \quad \text{(initial error)}$$

Propagated error

not a typo... why

$$e_{abs}((5.18+6.29)-1.31) \leq e_{abs}(5.18+6.29) + e_{abs}(1.31) = 0.01 + 0.005$$

We have this bound:

$$e_{abs}(x-y) = |x-y-(x^*-y^*)|$$

$$\leq |x-x^*| + |-y+y^*|$$

$$= |x-x^*| + |y-y^*| = e_{abs}(x) + e_{abs}(y)$$

Propagated error...

$$e_{abs}((5.18+6.29)-1.31) \leq 0.015.$$

Next: Generated error... (work with 3-digit arithmetic)

$$\begin{array}{r} 5.18 \\ +\ 6.29 \\ \hline 11.47 \end{array} \xrightarrow{\text{rounding}} 11.5$$

$$e_{gen} = |11.47 - 11.5| = .03$$

$$\begin{array}{r} 11.\cancel{5}^{4}0 \\ -\ 1.31 \\ \hline 10.19 \end{array} \xrightarrow{\text{rounding}} 10.2$$

$$e_{gen} = |10.19 - 10.2| = .01$$

For the whole computation

$$e_{gen} \leq 0.03 + 0.01 = 0.04$$

Accumulated error

$$e_{accumulated} = e_{propagated} + e_{generated}$$

$$\leq 0.015 + 0.04 = 0.055$$