Example: 3.55×2.73

If $x^* = 3.55$? then $x \in (3.545, 3.555)$

$y^* = 2.73$ then $y \in (2.725, 2.735)$

Use interval arithmetic to find the bounds on xy.

$$xy \in (9.660125, 9.722925)$$

the exact product has to lie in this interval

```
julia> 3.545*2.725
9.660125

julia> 3.555*2.735
9.722925
```

Last time we got

$$xy \in (9.6601, 9.7229)$$

which is almost the same... because the rule $e_{rel}(xy) \leq e_{rel}(x) + e_{rel}(y)$ is only approximately true, up to the higher order terms.

Generated and accumulated errors in this calculation... 3.55×2.73

working with 3S arithmetic and rounding.

```
julia> 3.55*2.73
9.6915
```

Therefore $x^* y^* = 9.6915 \xrightarrow{\text{round}} 9.69$

```
julia> 9.6915-9.69
0.0015000000000000568
```

Generated error

$$e_{gen} = |9.6915 - 9.69| = 0.0015$$

Propagated error (from last lecture)

$$e_{abs}(xy) = |xy|\, e_{rel}(xy) \approx |x^* y^*|\, e_{rel}(xy) = 0.0314$$

Accumulated error

$$e_{acc}(xy) = e_{prop}(xy) + e_{gen}(xy) \le 0.0314 + 0.0015$$

$$e_{acc}(xy) \le 0.0329$$

```
julia> 0.0314+0.0015
0.0329
```

Scientific computation uses double precision floating point which has about 15 decimal digits of precision. That's because errors accumulate in computations so it's okay if half of them disappear, because there still a lot left.

Taylor's theorem STEP 5

assume f' exists...

Fundamental Theorem of Calculus.

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(t)\,dt$$

Thus

$$f(x) = \underbrace{f(x_0)}_{\text{approximation of } f(x)} + \underbrace{\int_{x_0}^{x} f'(t)\,dt}_{\text{error in the approximation}}$$

0th order approximation of $f(x)$

$$T_0(x) = f(x_0) \qquad R_0(x) = \int_{x_0}^{x} f'(t)\,dt$$

Use integration by part to find higher order approx.

$$\int_{x_0}^{x} u\,dv = uv\Big|_{x_0}^{x} - \int_{x_0}^{x} v\,du$$

$$\int_{x_0}^{x} f'(t)\,dt = (t+c)f'(t)\Big|_{x_0}^{x} - \int_{x_0}^{x} (t+c)f''(t)\,dt$$

$u = f'(t)$ $\qquad du = f''(t)\,dt$

$dv = dt$ $\qquad v = \int dt = t + C.$ const of integration...

Choose c to obtain cancellations in the term $(t+c)f'(t)\Big|_{x_0}^{x}$

want the term when I plug in $t = x$ to cancel

$$(t+c)f'(t)\Big|_{x_0}^{x} = (x+c)f'(x) - (x_0+c)f'(x_0)$$

let $c=-x$ so this term cancels...

Note x is the constant that appears in the limits of integration, the variable in this case is t.

Thus ...

$$(t+c)f'(t)\Big|_{x_0}^{x} = -(x_0-x)f'(x_0)$$

Consequently

$$f(x) = f(x_0) - (x_0-x)f'(x_0) - \int_{x_0}^{x}(t-x)f''(t)\,dt$$

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \int_{x_0}^{x}(x-t)f''(t)\,dt$$

1st order approximation

$$T_1(x) = f(x_0) + (x-x_0)f'(x_0)$$

error

$$R_1(x) = \int_{x_0}^{x}(x-t)f''(t)\,dt$$

Do integration by parts again.

$$\int_{x_0}^{x}(x-t)f''(t)\,dt = -\tfrac{1}{2}(x-t)^2 f''(t)\Big|_{x_0}^{x} - \int_{x_0}^{x} -\tfrac{1}{2}(x-t)^2 f'''(t)\,dt$$

$$u = f''(t) \qquad du = f'''(t)\,dt$$

$$dv = (x-t)\,dt \qquad v = -\tfrac{1}{2}(x-t)^2$$

$$\int_{x_0}^{x}(x-t)f''(t)\,dt = \tfrac{1}{2}(x-x_0)^2 f''(x_0) + \int_{x_0}^{x}\tfrac{1}{2}(x-t)^2 f'''(t)\,dt$$

Thus,

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{1}{2}(x-x_0)^2 f''(x_0) + \int_{x_0}^{x} \frac{1}{2}(x-t)^2 f'''(t)\,dt$$

2nd order approximation

$$T_2(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{1}{2}(x-x_0)^2 f''(x_0)$$

error

$$R_2(x) = \int_{x_0}^{x} \frac{1}{2}(x-t)^2 f'''(t)\,dt$$

Integrate by parts again ... Details?

3rd order approximation

$$T_3(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{1}{2}(x-x_0)^2 f''(x_0) + \frac{1}{3!}(x-x_0)^3 f'''(x_0)$$

error

$$R_3(x) = \int_{x_0}^{x} \frac{1}{3!}(x-t)^3 f^{(4)}(t)\,dt$$

In general

approx of $f(x)$

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!}(x-x_0)^k f^{(k)}(x_0)$$

Note $0! = 1$

$(x-x_0)^0 = 1$

error

$$R_n(x) = \int_{x_0}^{x} \frac{1}{n!}(x-t)^n f^{(n+1)}(t)\,dt$$

Since $f(x) = \underset{\text{approx.}}{T_n(x)} + \underset{\text{error}}{R_n(x)}$

$$f(x) \approx T_n(x)$$

and now bound the error:--.

$$|R_n(x)| = \left| \int_{x_0}^{x} \frac{1}{n!}(x-t)^n f^{(n+1)}(t)\,dt \right|$$

$$\leq \left| \int_{x_0}^{x} \frac{1}{n!}|x-t|^n \max\{|f^{(n+1)}(t)| : t \in I\}\,dt \right|$$

here $I = \begin{cases} [x_0, x] & \text{for } x_0 \leq x \\ [x, x_0] & \text{for } x \leq x_0 \end{cases}$

$$\leq \max_{x \in I} |f^{(n+1)}(t)| \left| \int_{x_0}^{x} \frac{1}{n!}|x-t|^n\,dt \right|$$

... since taking abs. outside anyway.

$$\leq \max_{x \in I} |f^{(n+1)}(t)| \left| \int_{x_0}^{x} \frac{1}{n!}(x-t)^n\,dt \right|$$

$$\leq \max_{x \in I} |f^{(n+1)}(t)| \frac{1}{(n+1)!} \left| (x-x_0)^{n+1} \right|$$