$$f(\alpha) \approx f(x_n) + (\alpha - x_n) f'(x_n)$$

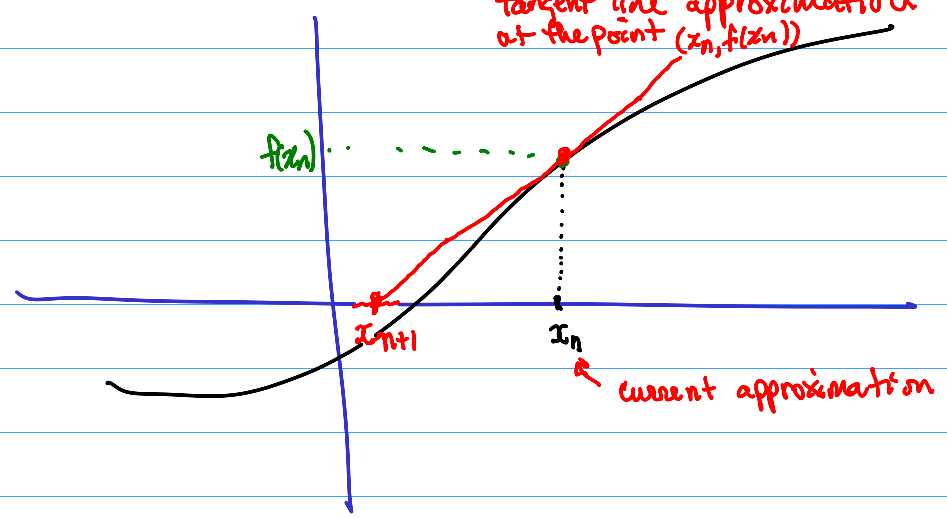
Newtons method

$$x_{n+1} \approx x_n - \frac{f(x_n)}{f'(x_n)}$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$x_{n+1} = g(x_n)$$

Derive it again geometrically...




Convergence of general iterative scheme. $x_{n+1} = g(x_n)$. Note that one example of such a iteration is Newton's method.

Suppose that $\alpha = g(\alpha)$ and that $x_n$ is supposed to approximate $\alpha$.

```
julia> xn=x0
2

julia> xn=xn-f(xn)/df(xn)
2.5221887802138703

julia> xn=xn-f(xn)/df(xn)
2.5425691666820014

julia> xn=xn-f(xn)/df(xn)
2.5426413568569757

julia> xn=xn-f(xn)/df(xn)
2.5426413577735265

julia> xn=xn-f(xn)/df(xn)
2.5426413577735265
```

These two were the same up to rounding errors. ... so Newton's method is really looking for a fixed point $\alpha$ such that $\alpha = g(\alpha)$.

Suppose that $\alpha = g(\alpha)$ and define $e_n = \alpha - x_n$

By the mean value theorem

$$e_1 = \alpha - x_1 = g(\alpha) - g(x_0) = (\alpha - x_0)\, g'(c_0) = e_0\, g'(c_0)$$

$e_1 = e_0\, g'(c_0)$ for some $c_0$ between $\alpha$ and $x_0$

$e_2 = e_1\, g'(c_1)$ for some $c_1$ between $\alpha$ and $x_1$.

$e_3 = e_2\, g'(c_2)$ for some $c_2$ between $\alpha$ and $x_2$.

Therefore

$$e_2 = e_0\, g'(c_0)\, g'(c_1)$$

$$e_3 = e_0\, g'(c_0)\, g'(c_1)\, g'(c_2)$$

By induction

$$e_{n+1} = e_0\, g'(c_0)\, g'(c_1) \cdots g'(c_n)$$

Let's suppose $|g'| < 1$ in a neighborhood of $\alpha$.

More specifically suppose there is $\delta > 0$ and $0 \le \gamma < 1$ such that

$$|g'(x)| \le \gamma \quad \text{for} \quad x \in (\alpha - \delta, \alpha + \delta)$$

Now if $x_0 \in (\alpha - \delta, \alpha + \delta)$ then $|e_0| = |\alpha - x_0| < \delta$

also $c_0$ which is between $\alpha$ and $x_0$ satisfies $c_0 \in (\alpha - \delta, \alpha + \delta)$.

$|e_1| = |e_0|\,|g'(c_0)| < \delta\gamma$ thus $x_1 \in (\alpha - \delta\gamma, \alpha + \delta\gamma)$

or simpler

$|e_1| = |e_0|\,|g'(c_0)| < \delta\gamma < \delta$ thus $x_1 \in (\alpha - \delta, \alpha + \delta)$

also $c_1$ which is between $\alpha$ and $x_1$ satisfies $c_1 \in (\alpha - \delta, \alpha + \delta)$.

$|e_2| = |e_1||g'(c_1)| \leq \delta\gamma < \delta$ thus $x_2 \in (\alpha-\delta, \alpha+\delta)$

Conclusion: if $x_0 \in (\alpha-\delta, \alpha+\delta)$ then $x_n \in (\alpha-\delta, \alpha+\delta)$ for all $n$.

then $c_n \in (\alpha-\delta, \alpha+\delta)$ for all $n$.

so $|g'(c_n)| \leq \gamma$ for all $n$.

Now

$$|e_{n+1}| = |e_0 g'(c_0) g'(c_1) \cdots g'(c_n)|$$
$$= |e_0| |g'(c_0)| |g'(c_1)| \cdots |g'(c_n)|$$
$$< \delta\, \gamma \cdot \gamma \cdots \gamma = \delta \gamma^{n+1} \to 0 \quad \text{as } n \to \infty$$

since $0 < \gamma < 1$.

---

(*) Theorem: Let $\alpha$ be the fixed point so $\alpha = g(\alpha)$.

Suppose $x_{n+1} = g(x_n)$

and $|g'(x)| \leq \gamma < 1$ for $x \in (\alpha-\delta, \alpha+\delta]$.

If $x_0 \in (\alpha-\delta, \alpha+\delta)$ then $x_n \to \alpha$ as $n \to \infty$.

---

Use this general result to study Newton's method:

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x) = 1 - \frac{f'(x) f'(x) - f(x) f''(x)}{(f'(x))^2} = \cancel{1} - \cancel{1} + \frac{f(x) f''(x)}{(f'(x))^2}$$

Therefore I want

$$|g'(x)| = \left| \frac{f(x)f''(x)}{(f'(x))^2} \right| \leq 1 \quad \text{for } x \text{ near } \alpha$$

(**) Theorem: Suppose $\alpha$ is the root so $f(\alpha)=0$ and $f'(\alpha)\neq 0$ and that $f$ is twice continuously differentiable, then provided $x_0$ is close enough to $\alpha$ then Newton's method converges.

Proof: What does close enough mean?

$$\left| \frac{f(x)f''(x)}{(f'(x))^2} \right| \leq 1 \quad \text{for } x \in (\alpha-\delta, \alpha+\delta)$$

or

$$|f(x)f''(x)| \leq 1\,|f'(x)|^2 \quad \text{for } x \in (\alpha-\delta, \alpha+\delta)$$

Plug in $\alpha$

$$|f(\alpha)f''(\alpha)| = |0 \cdot f''(\alpha)| = 0 < \underbrace{|f'(\alpha)|}^{2}$$

by hypothesis this is not zero.

Since

$$|f(\alpha)f''(\alpha)| < \underbrace{|f'(\alpha)|}^{2}$$

and $f, f'$ and $f''$ are continuous then there is $\delta > 0$ such that

$$|f(x)f''(x)| < \underbrace{|f'(x)|}^{2} \quad \text{for } x \in (\alpha-\delta, \alpha+\delta).$$

Then the Theorem (*) implies $x_n \to \alpha$.

Corollary: Suppose $\alpha$ is the root so $f(\alpha)=0$ and $f'(\alpha)\neq 0$ and that $f$ is twice continuously differentiable, then provided $x_0$ is close enough to $\alpha$ then Newton's method converges quadratically.

That is, from last friday

$$e_{n+1} \approx \frac{-e_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)}$$

so that

$$|e_{n+1}| \lesssim M|e_n|^2 \qquad \text{for all } n.$$