Forward error analysis of $Ax=b$ example...

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \pm e_1 \\ 1 \pm e_2 \end{bmatrix}$$

Augmented matrix $[A \mid b]$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \pm e_1 \\ 3 & 4 & 1 \pm e_2 \end{array}\right] \qquad r_2 \leftarrow r_2 - 3r_1$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \pm e_1 \\ 0 & -2 & -2 \pm (3e_1 + e_2) \end{array}\right] \qquad r_1 \leftarrow r_1 + r_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & -1 \pm (4e_1 + e_2) \\ 0 & -2 & -2 \pm (3e_1 + e_2) \end{array}\right] \qquad r_2 \leftarrow \frac{1}{-2} r_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & -1 \pm (4e_1 + e_2) \\ 0 & 1 & 1 \pm (\frac{3}{2}e_1 + \frac{1}{2}e_2) \end{array}\right]$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \pm (4e_1 + e_2) \\ 1 \pm (\frac{3}{2}e_1 + \frac{1}{2}e_2) \end{bmatrix}$$

Effect of partial pivoting...

$$\begin{bmatrix} 1 & 2 & 1 \pm e_1 \\ 3 & 4 & 1 \pm e_2 \end{bmatrix} \qquad r_1 \longleftrightarrow r_2$$

$$\begin{bmatrix} 3 & 4 & 1 \pm e_2 \\ 1 & 2 & 1 \pm e_1 \end{bmatrix} \qquad r_2 \leftarrow r_2 - \tfrac{1}{3} r_1$$

$$2 - \frac{4}{3} = \frac{6-4}{3} = \frac{2}{3}$$

$$\begin{bmatrix} 3 & 4 & 1 \pm e_2 \\ 0 & 2/3 & 2/3 \pm \left(e_1 + \frac{1}{3}e_2\right) \end{bmatrix} \qquad r_2 \leftarrow \tfrac{3}{2} r_2$$

$$\begin{bmatrix} 3 & 4 & 1 \pm e_2 \\ 0 & 1 & 1 \pm \left(\frac{3}{2}e_1 + \frac{1}{2}e_2\right) \end{bmatrix} \qquad r_1 \leftarrow r_1 - 4 r_2$$

$$\begin{bmatrix} 3 & 0 & -3 \pm (6e_1 + 3e_2) \\ 0 & 1 & 1 \pm \left(\frac{3}{2}e_1 + \frac{1}{2}e_2\right) \end{bmatrix} \qquad r_1 \leftarrow \tfrac{1}{3} r_1$$

$$\begin{bmatrix} 1 & 0 & -1 \pm (2e_1 + e_2) \\ 0 & 1 & 1 \pm \left(\frac{3}{2}e_1 + \frac{1}{2}e_2\right) \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \pm (2e_1 + e_2) \\ 1 \pm \left(\frac{3}{2}e_1 + \frac{1}{2}e_2\right) \end{bmatrix}$$

from before

more error here...

$$\begin{bmatrix} -1 \pm (4e_1 + e_2) \\ 1 \pm \left(\frac{3}{2}e_1 + \frac{1}{2}e_2\right) \end{bmatrix}$$

- If there were more rows, the effect of pivoting would be greater.
- The error bounds of x^* depend on the algorithm used..

Backwards error analysis. $Ax=b$. Let $x=\alpha$ be the exact solution and x^* be the approximation of α.

$$\left.\begin{aligned} A\alpha &= b \\ Ax^* &= b^* \end{aligned}\right\} \text{residual error } \|b-b^*\|$$

plug in the approximation.

$$e_{abs}(x^*) = \|\alpha - x^*\|.$$

Follow the book:

$$\begin{aligned} A\alpha &= b \\ -\ Ax^* &= b^* \\ \hline A(\alpha - x^*) &= b - b^* \end{aligned}$$

mult by A^{-1} to solve for the error in x^*.

$$\alpha - x^* = A^{-1}(b-b^*)$$

Thus

$$e_{abs}(x^*) = \|\alpha - x^*\| = \|A^{-1}(b-b^*)\| \leq \|A^{-1}\| \underbrace{\|b-b^*\|}_{\text{absolute residual error}}$$

If I know A^{-1} then why worry about x^* one could just compute $\alpha = A^{-1}b$. Even if I knew A^{-1} exactly $A^{-1}b$ likely has some rounding error.

How could I find $\|A^{-1}\|$ without knowing A^{-1}? This is possible and related to eigenvalues of A^TA or what's called the singular values of the matrix A.

What about relative error? (More appropriate way of characterizing error in floating point computations)

Recall.

$$\|\alpha - x^*\| \leq \|A^{-1}\| \|b - b^*\|$$

Note $\quad A\alpha = b$

$$e_{rel}(x^*) = \frac{\|\alpha - x^*\|}{\|\alpha\|} \qquad\qquad e_{rel}(b^*) = \frac{\|b - b^*\|}{\|b\|}$$

Estimate

$$\|b\| = \|A\alpha\| \leq \|A\| \|\alpha\| .$$

Thus,

$$\frac{1}{\|b\|} \geq \frac{1}{\|A\| \|\alpha\|}$$

or

$$\frac{1}{\|\alpha\|} \leq \frac{\|A\|}{\|b\|}$$

Now multiply the inequalities together to obtain

$$\frac{1}{\|\alpha\|} \|\alpha - x^*\| \leq \frac{\|A\|}{\|b\|} \|A^{-1}\| \|b - b^*\|$$

or

$$\boxed{\frac{\|\alpha - x^*\|}{\|\alpha\|} \leq \|A\| \|A^{-1}\| \frac{\|b - b^*\|}{\|b\|}}$$

This is the same as Monday's analysis of $f(x)$.

Recall from last time $f(x) = ax^2 + bx$, $f(\alpha) = c$, $f(x^*) = c^*$

$$f(\alpha) = f(x^*) + (\alpha - x^*) f'(\xi) \quad \text{for some } \xi \text{ between } \alpha \text{ and } x^*$$

$$f(\alpha) = f(0) + (\alpha - 0) f'(\eta) \quad \text{for some } \eta \text{ between } \alpha \text{ and } 0$$

$$c - c^* = (\alpha - x^*) f'(\xi) \quad \text{for some } \xi \text{ between } \alpha \text{ and } x^*$$

$$c = \alpha f'(\eta) \quad \text{for some } \eta \text{ between } \alpha \text{ and } 0$$

$$\frac{|c - c^*|}{|c|} = \frac{|f'(\xi)|}{|f'(\eta)|} \frac{|\alpha - x^*|}{|\alpha|}$$

$$\frac{|\alpha - x^*|}{|\alpha|} = \frac{|f'(\eta)|}{|f'(\xi)|} \frac{|c - c^*|}{|c|}$$

Compare to this

?

$$\frac{\|\alpha - x^*\|}{\|\alpha\|} \leq \|A\| \|A^{-1}\| \frac{\|b - b^*\|}{\|b\|}$$

Implicit function theorem: If $f'(\xi) \neq 0$ then f^{-1} exists in a neighborhood of ξ. In Calc I we do this

$$y = f(x) \quad \text{so} \quad x = f^{-1}(y)$$

$$\frac{dy}{dx} = f'(x) \qquad \frac{dx}{dy} = (f^{-1})'(y)$$

And

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad \boxed{\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}}$$

Thus

$$\boxed{(f^{-1})'(y) = \frac{1}{f'(x)}}$$

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

$$\frac{|\alpha - x^*|}{|\alpha|} = \frac{|f'(x)|}{|f'(\xi)|} \frac{|c - c^*|}{|c|}$$

$$\boxed{\frac{1}{f'(\xi)} = (f^{-1})'(f(\xi))}$$

Therefore,

$$\frac{|\alpha - x^*|}{|\alpha|} = |f'(x)| \, |(f^{-1})'(f(\xi))| \, \frac{|c - c^*|}{|c|}$$

$$\frac{\|\alpha - x^*\|}{\|\alpha\|} \leq \underbrace{\|A\| \|A^{-1}\|} \frac{\|b - b^*\|}{\|b\|}$$

this is called the condition number.

Take $f(x) = Ax$ $\quad f'(x) = A$

$f^{-1}(x) = A^{-1}x$ $\quad (f^{-1})'(x) = A^{-1}$

and they now match.