error in x^* residual error

$$\boxed{\frac{\|\alpha - x^*\|}{\|\alpha\|} \leq \underbrace{\|A\|\|A^{-1}\|}_{\substack{\text{condition number} \\ K(A) = \|A\|\|A^{-1}\|}} \frac{\|b-b^*\|}{\|b\|}}$$

Suppose b^* was computed using the standard floating point arithmetic: Means b^* has around 15 significant decimal digits, Thus

$$\frac{\|b - b^*\|}{\|b\|} \lesssim 5\times 10^{-15}$$

In the best case... under the assumption that b^* is as good as it can be...

Suppose

$$K(A) \approx 10^8 \qquad \leftarrow \text{very big...}$$

Then

$$\frac{\|\alpha - x^*\|}{\|\alpha\|} \lesssim 10^8 \cdot \tilde{5}\times 10^{-15} = 5\times 10^{-7}$$

So b^* is as good as it can be. This means that I can only verify 7 significant digits of x^* by plugging it in to the problem.

So it doesn't matter how I obtained x^*. The computer can't tell the difference between one and the other up to more than 7 significant digits.

The question then arises as to how large the condition number has to be for ill-conditioning to be a problem. Roughly speaking, if the condition number is 10^m and the machine being used to solve the linear system has k decimal digits of accuracy, then the solution of the linear system will be accurate to $k - m$ decimal digits. (at best).

If x^* is an approximation that minimizes the residual error, then it has 7 digits accuracy in the above case.

Matrix norms again:

use a different vector norm here, then I get a different matrix norm.

$$\underbrace{\|A\|}_{\text{matrix norm}} = \max\{\underbrace{\|Ax\|}_{\text{vector norm}} : \underbrace{\|x\|}_{\text{vector norm}} \leq 1\}$$

How do you find this maximum efficiently?

restrict to boundary gives the same maximum

$$\|A\| = \max\{\|Ax\| : \|x\| = 1\}$$

Geometry of the Euclidean norm:

$$\|x\| = \sqrt{x \cdot x} \qquad \|x\|^2 = x \cdot x$$

$$\|A\| = \left(\max\{\|Ax\|^2 : \|x\|^2 = 1\}\right)^{1/2}$$

Now

$$\|Ax\|^2 = Ax \cdot Ax = \underbrace{A^TA}_{B = A^TA} x \cdot x = Bx \cdot x$$

Note that B is symmetric ...

Spectral Theorem: If $B \in \mathbb{R}^{n \times n}$ and $B^T = B$. Then there exist a orthonormal basis for $\mathbb{R}^n$ made out of eigenvalues of B,

$$\boxed{\begin{array}{ll} B\xi_i = \lambda_i \xi_i \quad \text{and} & \xi_i \cdot \xi_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \\ \text{here } i = 1, \dots, n & i, j = 1, \dots, n. \end{array}}$$

Idea write the unit vector x in terms of the orthonormal basis of eigenvectors

$$x = c_1\xi_1 + c_2\xi_2 + \dots + c_n\xi_n \quad \text{for some } c_i \in \mathbb{R}.$$

$$= \sum_{i=1}^{n} c_i \xi_i$$

expand

Thus

$$1 = \|x\|^2 = x \cdot x = \sum_{i=1}^{n} c_i \xi_i \cdot \sum_{j=1}^{n} c_j \xi_j$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \underbrace{\xi_i \cdot \xi_j}_{\text{orthonormal}} = \sum_{i=1}^{n} c_i^2$$

$$\|Ax\|^2 = Bx \cdot x = B \sum_{i=1}^{n} c_i \xi_i \cdot \sum_{j=1}^{n} c_j \xi_j$$

$$= \sum_{i=1}^{n} c_i \underbrace{B \xi_i}_{\text{eigen vectors}} \cdot \sum_{j=1}^{n} c_j \xi_j \qquad B\xi_i = \lambda_i \xi_i$$

expand

$$= \sum_{i=1}^{n} c_i \lambda_i \xi_i \cdot \sum_{j=1}^{n} c_j \xi_j$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} c_i \lambda_i c_j \underbrace{\xi_i \cdot \xi_j}_{\text{orthonormal}} = \sum_{i=1}^{n} c_i^2 \lambda_i$$

$$\max\left\{ \|Ax\|^2 : \|x\|^2 = 1 \right\} = \max\left\{ \underbrace{\sum_{i=1}^{n} c_i^2 \lambda_i}_{\text{weighted average of the } \lambda_i\text{'s}} : \sum_{i=1}^{n} c_i^2 = 1 \right\}$$

Therefore,

$$\|A\| = \left(\max\{\lambda_i : i=1,\dots,n\}\right)^{1/2}$$

The average being less than the maximum value of what's being averages depends the λ_i's being real valued...

Recall

$$0 \leq \|A\xi_i\|^2 = A\xi_i \cdot A\xi_i = \underbrace{A^TA}_{}\xi_i \cdot \xi_i = B\xi_i \cdot \xi_i$$

$$= \lambda_i \xi_i \cdot \xi_i = \lambda_i$$

Therefore.

$$0 \leq \lambda_i$$

not only are the λ_i's real, but they are positive...

Consequently

$$\|A\| = \max\{\underbrace{\lambda_i^{1/2}}_{} : i=1,\dots,n\}.$$

$\sigma_i = \lambda_i^{1/2}$ are called the singular values of A.