Tables of differences...

```julia
f(x)=x^5
xs=0.1:0.01:0.18
N=length(xs)
M=zeros(N,N)
for i=1:N
    M[i,1]=f(xs[i])
end
for j=2:N
    for i=1:N-j+1
        M[i,j]=M[i+1,j-1]-M[i,j-1]
    end
end
```

5th degree polynomial...

propagated

If $f(x)$ is a polynomial of degree n then the nth differences are constant and the $(n+1)$-th differences are zero... except for rounding errors...

5th degree 5th differences are constant and the 6th differences and beyond are zero...

```
julia> M
9×9 Matrix{Float64}:
```

	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6		
1.0e-5	6.1051e-6	2.673e-6	7.95e-7	1.44e-7	1.2e-8	3.25261e-19	-1.01983e-18	2.37508e-18
1.61051e-5	8.7781e-6	3.468e-6	9.39e-7	1.56e-7	1.2e-8	-6.94567e-19	1.35525e-18	0.0
2.48832e-5	1.22461e-5	4.407e-6	1.095e-6	1.68e-7	1.2e-8	6.60686e-19	0.0	0.0
3.71293e-5	1.66531e-5	5.502e-6	1.263e-6	1.8e-7	1.2e-8	0.0	0.0	0.0
5.37824e-5	2.21551e-5	6.765e-6	1.443e-6	1.92e-7	0.0	0.0	0.0	0.0
7.59375e-5	2.89201e-5	8.208e-6	1.635e-6	0.0	0.0	0.0	0.0	0.0
0.000104858	3.71281e-5	9.843e-6	0.0	0.0	0.0	0.0	0.0	0.0
0.000141986	4.69711e-5	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.000188957	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Note that if you have a n^{th} degree polynomial then the n^{th} derivative is constant and the $(n+1)^{th}$ derivative zero...

for the same reasons...

① $E f_i = f_{i+1}$: $f_i = f(x_i) = f(x_0 + ih)$

② $\Delta f_i = E f_i - f_i$ $\quad \Delta = E - I$ $\quad f_\alpha = f(x_0 + \alpha h)$

non integer value...

③ $\nabla f_i = (I - E^{-1}) f_i$

④ $\delta f_i = (E^{1/2} - E^{-1/2}) f_i = f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}$.

What does $E^{1/2}$ really mean? It's the operation such that

$$E^{1/2} E^{1/2} = E$$

defines $E^{1/2}$ as anything for which this holds

$$E^{1/2} f_i = f_{i+1/2}$$

$$E^{1/2} E^{1/2} f_i = E^{1/2} f_{i+1/2} = f_{i+\frac{1}{2}+\frac{1}{2}} = f_{i+1} = E f_i$$

Note that $W f_i = - f_{i+1/2}$

$$W f_i = - f_{i+1/2}$$

$$W W f_i = W(-f_{i+1/2}) = -(-f_{i+\frac{1}{2}+\frac{1}{2}}) = f_{i+1} = E f_i$$

not so useful, though...

Instead use positive roots... take limits define

$$E^\alpha f_i = f_{i+\alpha} = f(x_0 + (i+\alpha)h) \quad \text{for any } \alpha \in \mathbb{R}.$$

Taylor's series...

$$E^{\theta} f_0 = f(x_0 + \theta h) = f(x_0) + \theta h f'(x_0) + \frac{(\theta h)^2}{2} f''(x_0) + \cdots$$

$\Delta = E - I$ $\qquad$ $E = I + \Delta$

$$f(x_0 + \theta h) = \underbrace{(I+\Delta)^{\theta}}_{\text{what does this really mean?}} f_0$$

Calculus

$$(1+x)^2 = 1 + 2x + x^2$$
$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$
$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$
$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

Pascal's triangle...

Binomial coefficients

$$(1+x)^5 = \binom{5}{0} + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5$$

Generalized Binomial theorem

$$(1+x)^{\alpha} = \binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots$$ infinite series (Taylor)

Where $\binom{\alpha}{0} = 1$ and $\binom{\alpha}{k} = \dfrac{(\alpha)(\alpha-1)\cdots(\alpha-(k-1))}{1 \cdot 2 \cdot \ldots \cdot k}$

$$\binom{5}{2} = \frac{(5)(5-1)}{1 \cdot 2} = \frac{20}{2} = 10$$

$$(1+x)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k$$

Taylor series expansion...

$$\Delta f_i = f_{i+1} - f_i$$

$$(I+\Delta)^2 = I + 2\Delta + \Delta^2$$

$$(I+\Delta)^2 f_i = (I + 2\Delta + \Delta^2) f_i = f_i + 2\Delta f_i + \Delta^2 f_i$$

Same algebra and Calculus.

$$= f_i + 2(f_{i+1} - f_i) + \Delta(f_{i+1} - f_i)$$

$$= f_i + 2f_{i+1} - 2f_i + (f_{i+2} - f_{i+1}) - (f_{i+1} - f_i)$$

$$= f_i + 2f_{i+1} - 2f_i + f_{i+2} - f_{i+1} - f_{i+1} + f_i$$

$$= (1-2+1) f_i + (2-1-1) f_{i+1} + f_{i+2} = f_{i+2} = E^2 f_i$$

$$E^\alpha f_i = (I+\Delta)^\alpha f_i = \sum_{k=0}^{\infty} \binom{\alpha}{k} \Delta^k f_i$$

terms from difference table.

$$E^\alpha f_0 = f(x_0 + \alpha h) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \Delta^k f_0$$

$$f(x) = x^5$$

$$E^\alpha f_0 = \sum_{k=0}^{5} \binom{\alpha}{k} \Delta^k f_0$$

interpolate $\frac{2}{3}$ of the way between 2 entries in the table...

$\alpha = \frac{2}{3}$ $\quad h = 0.01 \quad x_0 = 0.1$

$$E^{\alpha} f_0 = f\left(0.1 + \frac{2}{3}\, 0.01\right) = f(0.10\overline{6}) = 1.380840823\ldots$$

```
julia> 0.1+(2/3)*0.01
0.10666666666666667
```

```
julia> f(0.1+(2/3)*0.01)
1.380840823045268e-5
```

$$E^{\alpha} f_0 = \binom{\alpha}{0} f_0 + \binom{\alpha}{1}\Delta f_0 + \binom{\alpha}{2}\Delta^2 f_0 + \binom{\alpha}{3}\Delta^3 f_0 + \binom{\alpha}{4}\Delta^4 f_0 + \binom{\alpha}{5}\Delta^5 f_0$$

$$= f_0 + \alpha \Delta f_0 + \frac{\alpha(\alpha-1)}{2}\Delta^2 f_0 + \frac{\alpha(\alpha-1)(\alpha-2)}{6}\Delta^3 f_0$$

$$+ \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!}\Delta^4 f_0 + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{5!}\Delta^5 f_0$$

polynomial in α

We'll use this later for interpolation...

```
julia> p(2/3)
1.3808408230452637e-5
```

agrees up to rounding

If f is not a polynomial then this would be a polynomial approximation of $f(x_0 + \alpha h)$ in α.

```
function p(alpha)
    b=1.0
    s=0
    for j=1:N
        s+=b*M[1,j]
        b*=(alpha-j+1)/j
    end
    return s
end
```