Recall

$$f_i = f(x_0 + ih), \qquad h > 0$$

$$E f_i = f_{i+1}$$

$$\Delta f_i = f_{i+1} - f_i = (E - I) f_i \qquad \Delta = E - I \qquad E = \Delta + I$$

$$\nabla f_i = f_i - f_{i-1} = (I - E^{-1}) f_i \qquad \nabla = I - E^{-1}$$

$$\nabla = I - E^{-1}, \quad E^{-1} = I - \nabla, \quad E = (I - \nabla)^{-1}$$

for some $\alpha \in (0,1)$

Interpolation

$$f(x_i + \alpha h) = E^{\alpha} f_i = (I + \Delta)^{\alpha} f_i$$

a value of f off the grid...

binomal theorem to expand this

having abstracted the idea of shifting the argument as an operator allows, to work algebraically with operators. In particular to use the binomial theorem

$$= \sum_{k=0}^{\infty} \binom{\alpha}{k} \Delta^k f_i$$

there are points in the table of differences based on grid points x_i

If f is a polynomial of degree n then

then the nth differences are constant
and the $(n+1)$-th differences are zero... except for rounding errors...

$$f(x_i+\alpha h) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \Delta^k f_i$$

if $\Delta^k f_i = 0$ for $k > n$

$$f(x_i+\alpha h) = \sum_{k=0}^{n} \binom{\alpha}{k} \Delta^k f_i$$

If f is not a polynomial, then

$$f(x_i+\alpha h) \approx \sum_{k=0}^{n} \binom{\alpha}{k} \Delta^k f_i$$

is an approximation using a degree n polynomial in α on the right hand side.

That's because the binomial coefficients are polynomials in α.

$$\binom{\alpha}{0} = 1 \quad \text{and} \quad \binom{\alpha}{k} = \frac{\alpha \cdot (\alpha-1) \cdots (\alpha-(k-1))}{1 \cdot 2 \cdots k}$$

The term $\alpha \cdot (\alpha-1) \cdots (\alpha-(k-1))$ occurs frequently to have a name in mathematics.

In mathematics, the **falling factorial** (sometimes called the **descending factorial**,[1] **falling sequential product**, or **lower factorial**) is defined as the polynomial

$$\begin{aligned}(x)_n = x^{\underline{n}} &= \overbrace{x(x-1)(x-2)\cdots(x-n+1)}^{n \text{ factors}} \\ &= \prod_{k=1}^{n}(x-k+1) = \prod_{k=0}^{n-1}(x-k).\end{aligned}$$

The **rising factorial** (sometimes called the **Pochhammer function**, **Pochhammer polynomial**, **ascending factorial**,[1] **rising sequential product**, or **upper factorial**) is defined as

$$\begin{aligned}x^{(n)} = x^{\overline{n}} &= \overbrace{x(x+1)(x+2)\cdots(x+n-1)}^{n \text{ factors}} \\ &= \prod_{k=1}^{n}(x+k-1) = \prod_{k=0}^{n-1}(x+k).\end{aligned}$$

$$f(x_i+\alpha h) \approx \sum_{k=0}^{n} \binom{\alpha}{k} \Delta^k f_i = \underbrace{\sum_{k=0}^{n} \frac{(\alpha)_k}{k!}}\Delta^k f_i$$

polynomial of degree n in α

so N=n-1 here

```
function p(alpha)
    b=1.0
    s=0
    for j=1:N
        s+=b*M[1,j]
        b*=(alpha-j+1)/j
    end
    return s
end
```

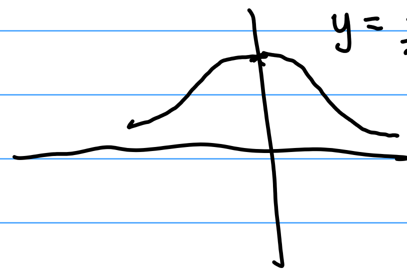
last term is M[1,N]

this is $\Delta^{N-1} f_0$

julia> M
9×9 Matrix{Float64}:

1	2 Δ	3 Δ^2	4 Δ^3	Δ^4	Δ^5	Δ^6		
1.0e-5	6.1051e-6	2.673e-6	7.95e-7	1.44e-7	1.2e-8	3.25261e-19	-1.01983e-18	2.37508e-18
1.61051e-5	8.7781e-6	3.468e-6	9.39e-7	1.56e-7	1.2e-8	-6.94567e-19	1.35525e-18	0.0
2.48832e-5	1.22461e-5	4.407e-6	1.095e-6	1.68e-7	1.2e-8	6.60686e-19	0.0	0.0
3.71293e-5	1.66531e-5	5.502e-6	1.263e-6	1.8e-7	1.2e-8	0.0	0.0	0.0
5.37824e-5	2.21551e-5	6.765e-6	1.443e-6	1.92e-7	0.0	0.0	0.0	0.0
7.59375e-5	2.89201e-5	8.208e-6	1.635e-6	0.0	0.0	0.0	0.0	0.0
0.000104858	3.71281e-5	9.843e-6	0.0	0.0	0.0	0.0	0.0	0.0
0.000141986	4.69711e-5	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.000188957	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

$$f(x) = \frac{1}{x^2+1}$$



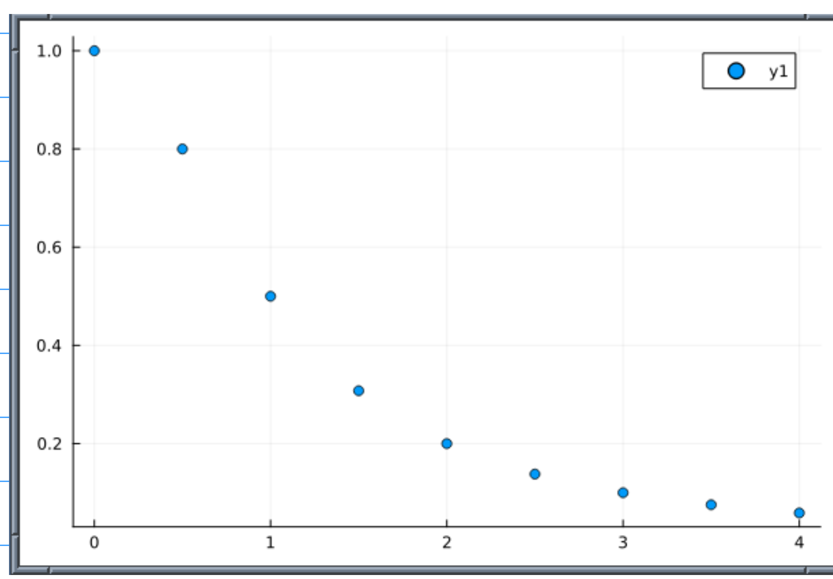
$$y = \frac{1}{x^2+1}$$

```
julia> f(x)=1/(x^2+1)
f (generic function with 1 method)

julia> xs=0:0.5:4
0.0:0.5:4.0

julia> using Plots

julia> scatter(xs,f.(xs))
```



Create the table of differences:

```
julia> N=length(xs)
       M=zeros(N,N)
       for i=1:N
           M[i,1]=f(xs[i])
       end
       for j=2:N
           for i=1:N-j+1
               M[i,j]=M[i+1,j-1]-M[i,j-1]
           end
       end
```

Note: none of the differences vanish to zero since this is not a polynomial.

```
julia> M
9×9 Matrix{Float64}:
 1.0        -0.2        -0.1         0.207692   -0.230769    0.214854    -0.181432   0.141254  -0.0998572
 0.8        -0.3         0.107692   -0.0230769  -0.0159151   0.0334218   -0.0401782  0.041397   0.0
 0.5        -0.192308    0.0846154  -0.038992    0.0175066  -0.00675642   0.0012188  0.0        0.0
 0.307692   -0.107692    0.0456233  -0.0214854   0.0107502  -0.00553761   0.0        0.0        0.0
 0.2        -0.062069    0.0241379  -0.0107352   0.0052126   0.0          0.0        0.0        0.0
 0.137931   -0.037931    0.0134027  -0.0055226   0.0         0.0          0.0        0.0        0.0
 0.1        -0.0245283   0.00788013  0.0         0.0         0.0          0.0        0.0        0.0
 0.0754717  -0.0166482   0.0         0.0         0.0         0.0          0.0        0.0        0.0
 0.0588235   0.0         0.0         0.0         0.0         0.0          0.0        0.0        0.0
```

We'll approximate this function using the first 4 differences to create a polynomial of degree 4 passing through the first 5 points.

```
julia> M
9×9 Matrix{Float64}:
```

	Δ	Δ^2	Δ^3	Δ^4
1.0	-0.2	-0.1	0.207692	-0.230769
0.8	-0.3	0.107692	-0.0230769	-0.0159151
0.5	-0.192308	0.0846154	-0.038992	0.0175066
0.307692	-0.107692	0.0456233	-0.0214854	0.0107502
0.2	-0.062069	0.0241379	-0.0107352	0.0052126
0.137931	-0.037931	0.0134027	-0.0055226	0.0
0.1	-0.0245283	0.00788013	0.0	0.0
0.0754717	-0.0166482	0.0	0.0	0.0
0.0588235	0.0	0.0	0.0	0.0

These 5 values of f are used to make the polynomial of degree 4

```julia
function p(alpha,d)
    b=1.0
    s=0
    for j=1:d+1
        s+=b*M[1,j]
        b*=(alpha-j+1)/j
    end
    return s
end
```

polynomial of degree d approximation of α

$$f(0+\alpha(0.5)) \approx p(\alpha, d)$$

$$x = \alpha(0.5)$$

$$\alpha = 2x$$

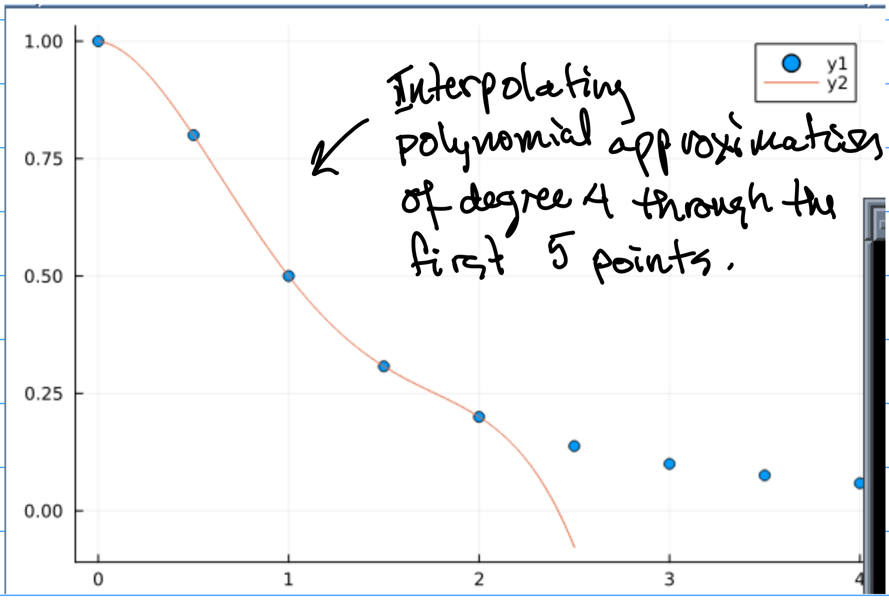
approximation $g(x) \approx f(x)$

```
julia> g(x)=p(2*x,4)
g (generic function with 1 method)
```

```
julia> xs2=0:0.01:2.5
0.0:0.01:2.5

julia> scatter(xs,f.(xs))

julia> plot!(xs2,g.(xs2))
```



Interpolating polynomial approximation of degree 4 through the first 5 points.

Note that one could solve for the polynomial coefficients directly

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

and then solve the linear system of 5 equations

$$\begin{cases} p(x_0) = f(x_0) \\ p(x_1) = f(x_1) \\ \vdots \\ p(x_4) = f(x_4) \end{cases}$$

$$\begin{cases} a_0 + a_1 x_0 + a_2 x_0^2 + a_3 x_0^3 + a_4 x_0^4 = f(x_0) \\ a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 + a_4 x_1^4 = f(x_1) \\ a_0 + a_1 x_4 + a_2 x_4^2 + a_3 x_4^3 + a_4 x_4^4 = f(x_4) \end{cases}$$

much more difficult to solve for the a's

It easier using the binomial theorem like we did...

we have a different polynomial basis that makes it easier to find and evaluate the polynomial.

$\binom{\alpha}{k}$ is a polynomial of degree k