$$f(x_i+\alpha h) \approx \sum_{k=0}^{\overset{n=4}{m}} \binom{\alpha}{k} \Delta^k f_i$$

```
julia> M
9×9 Matrix{Float64}:
```

x_i		Δ	Δ^2	Δ^3	Δ^4 (d=4)
0	1.0	-0.2	-0.1	0.207692	-0.230769
0.5	0.8	-0.3	0.107692	-0.0230769	-0.0159151
1.0	0.5	-0.192308	0.0846154	-0.038992	0.0175066
1.5	0.307692	-0.107692	0.0456233	-0.0214854	0.0107502
2	0.2	-0.062069	0.0241379	-0.0107352	0.0052126
2.5	0.137931	-0.037931	0.0134027	-0.0055226	0.0
3	0.1	-0.0245283	0.00788013	0.0	0.0
3.5	0.0754717	-0.0166482	0.0	0.0	0.0
4.0	0.0588235	0.0	0.0	0.0	0.0

```julia
function p(alpha,d)
    b=1.0
    s=0
    for j=1:d+1
        s+=b*M[1,j]
        b*=(alpha-j+1)/j
    end
    return s
end
```

d=4

i	x_i
0	0
1	0.5
2	1.0
3	1.5
4	2
5	2.5
6	3
	3.5
	4.0

```
julia> M
9×9 Matrix{Float64}:
 1.0        -0.2          -0.1          0.207692   -0.230769
 0.8        -0.3           0.107692    -0.0230769  -0.0159151
 0.5        -0.192308      0.0846154   -0.038992    0.0175066
 0.307692   -0.107692      0.0456233   -0.0214854   0.0107502
 0.2        -0.062069      0.0241379   -0.0107352   0.0052126
 0.137931   -0.037931      0.0134027   -0.0055226   0.0
 0.1        -0.0245283     0.00788013   0.0         0.0
 0.0754717  -0.0166482     0.0          0.0         0.0
 0.0588235   0.0           0.0          0.0         0.0
```

interpolating polynomial through these points

$$f(x_1+\alpha h) \approx \sum_{k=0}^{4} \binom{\alpha}{k} \Delta^k f_1 = p(\alpha)$$

$$x = x_1 + \alpha h, \quad \alpha = \frac{x-x_1}{h}$$

$$f(x) \approx p\left(\frac{x-x_1}{h}\right)$$

$$x_1 = 0.5$$

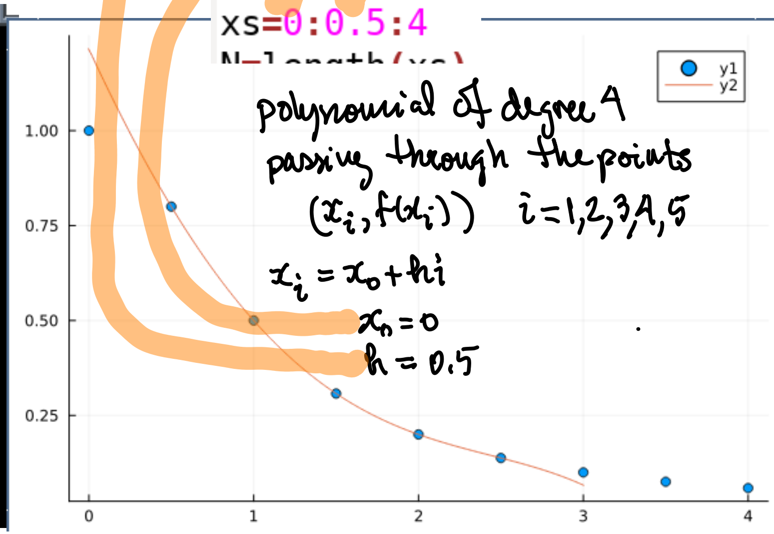
$$f(x) \approx p\left(\frac{x-0.5}{0.5}\right)$$

```
function p(alpha,d)
    b=1.0
    s=0
    for j=1:d+1
        s+=b*M[2,j]
        b*=(alpha-j+1)/j
    end
    return s
end
```

Change to 2, to start with 2nd point

```
julia> using Plots

julia> scatter(xs,f.(xs))

julia> xs2=0:0.01:3
0.0:0.01:3.0

julia> g(x)=p((x-0.5)/0.5,4)
g (generic function with 1 method)

julia> plot!(xs2,g.(xs2))
```

```
xs=0:0.5:4
```



polynomial of degree 4 passing through the points $(x_i, f(x_i))$ $i=1,2,3,4,5$

$x_i = x_0 + hi$

$x_0 = 0$

$h = 0.5$

Recall finite difference operators:

$$E f_i = f_{i+1}$$

$$\Delta f_i = f_{i+1} - f_i$$ ← appears in table of differences

$$\boxed{\nabla f_i = f_i - f_{i-1}}$$ ← in the same table but have to read it differently..

$$\delta f_i = f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}$$

focus on this one ...

$$\nabla = I - E^{-1}$$ solve for E.

$$E^{-1} = I - \nabla$$

$$E = (I - \nabla)^{-1}$$

$$f(x_i + \alpha h) = E^{\alpha} f_i = (I - \nabla)^{-\alpha} f_i = \sum_{k=0}^{\infty} \binom{-\alpha}{k} (-\nabla)^k f_i$$

binomial theorem

Approximation

$$f(x_i + \alpha h) \approx \sum_{k=0}^{n} \binom{-\alpha}{k} (-\nabla)^k f_i$$

where

$$\binom{-\alpha}{0} = 1 \quad \text{and} \quad \binom{-\alpha}{k} = \frac{(-\alpha)(-\alpha-1) \cdots (-\alpha-(k-1))}{1 \cdot 2 \cdot \cdots \cdot k}$$

Therefore

$$\binom{-\alpha}{k}(-\nabla)^k = \frac{\overbrace{(-\alpha)(-\alpha-1) \cdots (-\alpha-(k-1))}^{k \text{ terms}}}{1 \cdot 2 \cdot \cdots \cdot k} (-1)^k \nabla^k$$

k negatives here

$$= \frac{(\alpha)(\alpha+1) \cdots (\alpha+(k-1))}{1 \cdot 2 \cdots k} \nabla^k = \frac{\alpha^{(k)}}{k!} \nabla^k$$

rising factorial

recall the rising factorial is

$$x^{(n)} = x^{\overline{n}} = \overbrace{x(x+1)(x+2)\cdots(x+n-1)}^{n \text{ factors}}$$
$$= \prod_{k=1}^{n}(x+k-1) = \prod_{k=0}^{n-1}(x+k).$$

Note polynomials of degree n passing through $n+1$ points are uniquely determined, so should get the same polynomial with this formula as the other.

We are expressing the polynomial in terms of $\binom{-\alpha}{k}$ which is a different polynomial basis than $\binom{\alpha}{k}$ for $k=0,1,\dots,n$.

i	x_i
0	0
1	0.5
2	1
3	1.5
4	2
5	2.5

```
julia> M
9×9 Matrix{Float64}:
 1.0         -0.2         -0.1
 0.8         -0.3          0.107
 0.5         -0.192308     0.084
 0.307692    -0.107692     0.045
 0.2         -0.062069     0.024
 0.137931    -0.037931     0.013
 0.1         -0.0245283    0.007
 0.0754717   -0.0166482    0.0
 0.0588235    0.0          0.0
```

$\nabla f_4 =$ -0.107692 $\nabla^2 f_5 =$ 0.045

$\nabla f_5 =$ -0.062069

The backwards differences $\nabla^k f_6$ lie on this ascending diagonal.

$i=5$, $\quad \nabla f_5 = f_5 - f_4$

$$\nabla^2 f_5 = \nabla f_5 - \nabla f_4 = f_5 - f_4 - (f_4 - f_3)$$

$$f(x_5+\alpha h) \approx \sum_{k=0}^{4} \binom{-\alpha}{k} (-\nabla)^k f_5 = p(\alpha)$$

$$x = x_5 + \alpha h \qquad \alpha = \frac{x - x_5}{h} = \frac{x - 2.5}{0.5}$$

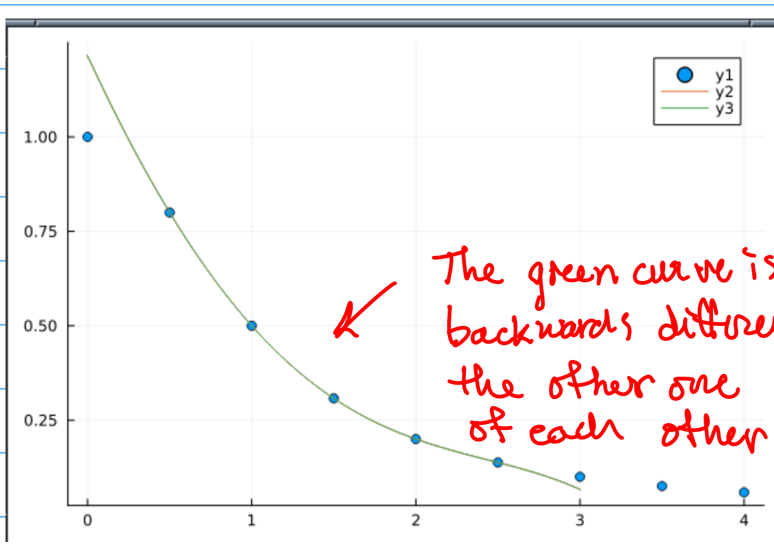
$$f(x) \approx p\left(\frac{x-2.5}{0.5}\right)$$

```
function p(alpha,d)
    b=1.0
    s=0
    for j=1:d+1
        s+=b*M[7-j,j]
        b*=(alpha+j-1)/j
    end
    return s
end
```

backward differences for f_5

$j=1$, $7-j=6$ and $M[6,1]=f_5$

```
julia> include("table.jl")
p (generic function with 1 method)

julia> gb(x)=p((x-2.5)/0.5,4)
gb (generic function with 1 method)

julia> plot!(xs2,gb.(xs2))
```



The green curve is the polynomial created by backwards differences and its the same as the other one because the curves are on top of each other..