Step 23 Lagrange Interpolating Polynomials

Review

coef of the basis ... from the difference table

$$f(x_i + \alpha h) \approx \sum_{k=0}^{n} \binom{\alpha}{k} \Delta^k f_i \qquad \text{polynomial of degree } n \text{ in } \alpha$$

polynomial basis functions

coef of the basis ... from the difference table

$$f(x_i + \alpha h) \approx \sum_{k=0}^{n} \binom{-\alpha}{k} (-\nabla)^k f_i$$

What about any other polynomial basis?

General problem: Find the polynomial passing through the points

$$(x_0, f(x_0)),\ (x_1, f(x_1)),\ \dots,\ (x_n, f(x_n))$$

where x's are all different. Theory from algebra. $n+1$ points uniquely determine a polynomial of at most degree n.

General basis for polynomials of degree n.

$$\phi_0,\ \phi_1,\ \dots,\ \phi_n$$

Then

$$p(x) = a_0\phi_0(x) + a_1\phi_1(x) + \cdots + a_n\phi_n(x)$$

The standard polynomial basis

$$\phi_0(x) = 1,\quad \phi_1(x) = x,\quad \phi_2(x) = x^2,\ \dots,\ \phi_n(x) = x^n$$

Then

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

Solve so the polynomial passes through the points

$$\begin{cases} p(x_0) = f(x_0) \\ p(x_1) = f(x_1) \\ \vdots \\ p(x_4) = f(x_4) \end{cases}$$

Linear equations

a is the vector

$$a_0 + a_1 x_0 + a_2 x_0^2 + \dots + a_n x_0^n = f(x_0)$$
$$a_0 + a_1 x_1 + a_2 x_1^2 + \dots + a_n x_1^n = f(x_1)$$
$$\vdots$$
$$a_0 + a_1 x_n + a_2 x_n^2 + \dots + a_n x_n^n = f(x_n)$$

In matrix form

matrix what multiplies the vector a

$$M = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & & x_n^n \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

Solve the matrix equation for a:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}$$

Vandermonde matrix: linear algebra result say M is invertible.

Same result as there being a unique polynomial..!

In terms of general basis functions ϕ_i what does this matrix equation look like?

n+1 linear equations in n+1 unknowns

$$\begin{cases} p(x_0) = a_0\phi_0(x_0) + a_1\phi_1(x_0) + \cdots + a_n\phi_n(x_0) = f(x_0) \\ p(x_1) = a_0\phi_0(x_1) + a_1\phi_1(x_1) + \cdots + a_n\phi_n(x_1) = f(x_1) \\ \vdots \\ p(x_n) = a_0\phi_0(x_n) + a_1\phi_1(x_n) + \cdots + a_n\phi_n(x_n) = f(x_n) \end{cases}$$

Since the basis functions ϕ_i are linearly independent, then the columns of the matrix are independent

$$M = \begin{bmatrix} \phi_0(x_0) & \phi_1(x_0) & \cdots & \phi_n(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \cdots & \phi_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(x_n) & \phi_1(x_n) & \cdots & \phi_n(x_n) \end{bmatrix}$$

and $Ma = \begin{bmatrix} f(x_0) \\ \vdots \\ f(x_n) \end{bmatrix}$ has a unique solution...

What choice of basis functions make the matrix M easy to invert?

Lagrange basis functions is a polynomial basis that make $M = I$ which is super easy to invert.

True/False: The basis made with binomial coefficients $\binom{\alpha}{k}$ result in a triangular matrix? Upper or lower?

What I want $\begin{bmatrix} \varphi_0(x_0) & \varphi_1(x_0) & \cdots & \varphi_n(x_0) \\ \varphi_0(x_1) & \varphi_1(x_1) & \cdots & \varphi_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_0(x_n) & \varphi_1(x_n) & \cdots & \varphi_n(x_n) \end{bmatrix} = I$

$$\varphi_j(x_k) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$ and φ_j is a polynomial.

Note $\varphi_j(x_k) = 0$ when $k \neq j$ means $(x - x_k)$ is a factor of φ_j.
for all $k \neq j$ means $(x - x_k)$ is a factor for all $k \neq j$

$$h_j(x) = \prod_{i \neq j} (x - x_i)$$

$$h_j(x_k) = \prod_{i \neq j} (x_k - x_i) = \begin{cases} 0 & \text{if } k \neq j \\ \prod_{i \neq j} (x_j - x_i) \neq 0 & \text{if } k = j \end{cases}$$

$$L_j(x) = \frac{h_j(x)}{h_j(x_j)} = \frac{\prod_{i \neq j} (x - x_i)}{\prod_{i \neq j} (x_j - x_i)}$$

Thus

$$\begin{bmatrix} L_0(x_0) & L_1(x_0) & \cdots & L_n(x_0) \\ L_0(x_1) & L_1(x_1) & \cdots & L_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ L_0(x_n) & L_1(x_n) & \cdots & L_n(x_n) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}$$

becomes

$$I\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}$$

So

$$p(x) = a_0 h_0(x) + a_1 h_1(x) + \cdots + a_n h_n(x)$$

$$\boxed{p(x) = f(x_0) h_0(x) + f(x_1) h_1(x) + \cdots + f(x_n) h_n(x)}$$

Lagrange interpolating polynomial...

Good the coefficients are even easier than writing out the table of differences,

Bad There is no table of differences that can be used to judge the quality of the interpolation.

Good The values of x_i don't need to be equally spaced

Example: Find the polynomial through

$(1,3)$, $(2,5)$, $(4,-7)$

$x_0=1,\ f(x_0)=3$ $\quad x_1=2,\ f(x_1)=5$ $\quad x_2=4,\ f(x_2)=-7$

In a table

i	x_i	f_i
0	1	3
1	2	5
2	4	-7

Because the x's aren't equally spaced can't make a difference table... but Lagrange basis works

$$h_0(x) = \frac{\prod_{i\neq 0}(x-x_i)}{\prod_{i\neq 0}(x_0-x_i)} = \frac{(x-2)(x-4)}{(1-2)(1-4)}$$

$$h_1(x) = \frac{\prod_{i\neq 1}(x-x_i)}{\prod_{i\neq 1}(x_1-x_i)} = \frac{(x-1)(x-4)}{(2-1)(2-4)}$$

$$h_2(x) = \frac{\prod_{i\neq 2}(x-x_i)}{\prod_{i\neq 2}(x_2-x_i)} = \frac{(x-1)(x-2)}{(4-1)(4-2)}$$

Thus

$$p(x) = f_0 h_0(x) + f_1 h_1(x) + f_2 h_2(x)$$

$$= 3\frac{(x-2)(x-4)}{(1-2)(1-4)} + 5\frac{(x-1)(x-4)}{(2-1)(2-4)} - 7\frac{(x-1)(x-2)}{(4-1)(4-2)}$$

Simplify it... or not...