We want to factor $A = QR$

where Q is an orthogonal matrix
and R is upper triangular...

official definition
$Q^{-1} = Q^T$
equivalent to
$Q \in \mathbb{R}^{n\times n}$
and
$Q^T Q = I$

equations...
variables..

Note, in general $A \in \mathbb{R}^{m\times n}$ where $m >> n$. That's why the equation $\underline{Ax=b}$ can't be solved

overdetermined means more equations than variables

$$\underset{m\times n}{A} = \underset{m\times m}{Q}\,\underset{m\times n}{R}$$

$Q \in \mathbb{R}^{m\times m}$

$R \in \mathbb{R}^{m\times n}$

$$R = \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \\ \hline & 0 & \end{bmatrix} = \begin{array}{c} \leftarrow n \rightarrow \\ \begin{bmatrix} \tilde{R} \\ \hline 0_{m-n\times n} \end{bmatrix} \end{array}$$

$\tilde{R} \in \mathbb{R}^{n\times n}$

$Q_1 \in \mathbb{R}^{m\times n}$ $\qquad$ $Q_2 \in \mathbb{R}^{m\times(m-n)}$

$$\underset{m\times n}{A} = QR = \left[\, Q_1 \mid Q_2 \,\right] \begin{bmatrix} \tilde{R} \\ \hline 0 \end{bmatrix} = \underset{m\times n}{Q_1}\underset{n\times n}{\tilde{R}} + \underset{m\times(m-n)}{Q_2}\ \underset{(m-n)\times n}{0} = Q_1 \tilde{R}$$

Let $\tilde{Q} = Q_1$

Reduced QR decomposition

$$\underset{}{A} = \underset{m\times n}{\tilde{Q}}\ \underset{n\times n}{\tilde{R}}$$

$\tilde{R} \in \mathbb{R}^{n\times n}$ $\quad$ $\tilde{R}$ upper triangular

$\tilde{Q} \in \mathbb{R}^{m\times n}$ $\quad$ $\underset{n\times m}{\tilde{Q}^T}\ \underset{m\times n}{\tilde{Q}} = \underset{n\times n}{I}$

Note the reduced QR decomposition is what naturally comes from Gram-Schmidt..

$$\boxed{\underset{m\times n}{A} = \left[\, a_1 \mid a_2 \mid \cdots \mid a_n \,\right]} \qquad \underset{m\times n}{\tilde{Q}} = \left[\, q_1 \mid q_2 \mid \cdots \mid q_n \,\right]$$

$t_1 = a_1 \qquad\qquad q_1 = \frac{t_1}{\|t_1\|}$

$t_2 = a_2 - (q_1 \cdot a_2) q_1 \qquad\qquad q_2 = \frac{t_2}{\|t_2\|}$

$t_3 = a_3 - (q_1 \cdot a_3) q_1 - (q_2 \cdot a_3) q_2 \qquad\qquad q_3 = \frac{t_3}{\|t_3\|}$

$\vdots$

$t_n = a_n - (q_1 \cdot a_n) q_1 - \cdots - (q_{n-1} \cdot a_n) q_{n-1} \qquad\qquad q_n = \frac{t_n}{\|t_n\|}$

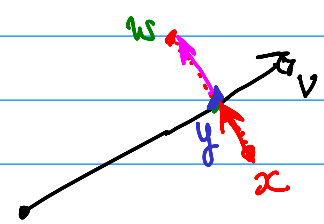
Given $A = \tilde{Q}\tilde{R}$ one can extend to the full QR decomposition as

$$R = \begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix} \qquad Q = \left[\begin{array}{c|c} \tilde{Q} & ? \end{array}\right]$$

anything here that is orthogonal to $\tilde{Q}$ is fine.

Our Goal is to construct Q directly and use the flexibility of it not being unique to reduce rounding errors..

Householder reflections... Given a unit vector $v \in \mathbb{R}^m$ then



Find w in terms of x and v

y = projection of x on v

$y = (v \cdot x)v$

$w - y = y - x$ (reflection)

$$w = w - y + y = y - x + y = 2y - x$$

$$= 2\underbrace{(v \cdot x)}_{\text{scalar}}v - x = 2v(v \cdot x) - x = 2vv^T x - x$$

$$= \underbrace{(2vv^T - I)}_{-H}x = -Hx$$

where $H = I - 2vv^T$.

Since reflections preserve lengths and angles we know H will be orthogonal. But lets check it.

$$H = I - 2vv^T \in \mathbb{R}^{m\times m}$$

Note $H^T = H$

$$H^TH = (I-2vv^T)^T(I-2vv^T)$$

$$= (I^T - (2vv^T)^T)(I-2vv^T)$$

$$= (I - 2v^{TT}v^T)(I-2vv^T)$$

$$= (I-2vv^T)(I-2vv^T) = I + 4v\cancel{v^Tv}v^T - 2vv^T - 2vv^T$$

$v^Tv = v\cdot v = |v|^2 = 1$

$$= I + \cancel{4vv^T} \cancel{- 2vv^T} \cancel{- 2vv^T} = I$$

Official definition

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equivalent to

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and

$Q^TQ = I$

<u>Idea:</u> Create QR decomposition by multiplying A on the left by H.

$$\underset{m\times m}{H}\;\underset{m\times n}{A} = \begin{bmatrix} Ha_1 & Ha_2 & \cdots & Ha_n \end{bmatrix} = \begin{bmatrix} \begin{matrix} c \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & Ha_2 & \cdots & Ha_n \end{bmatrix}$$

choose unit vector v for equality

Repeat this column by column

$$\underbrace{H_n \cdots H_2 H_1}_{\text{invert these}} A = R$$ ← upper triangular

$$A = (H_n \cdots H_2 H_1)^{-1} R$$

$$= H_1^{-1} H_2^{-1} \cdots H_n^{-1} R$$

$$= H_1^T H_2^T \cdots H_n^T R$$

$$= H_1 H_2 \cdots H_n R$$

Thus $A = QR$ where $Q = H_1 H_2 \cdots H_n$.

For the first step I need $Ha_1 = ce_1$

$$H = I - 2vv^T$$

so

$$(I - 2vv^T)a_1 = ce_1$$

solve for v...

$$a_1 - 2v\underbrace{v^T a_1}_{\text{dot product}} = ce_1$$

$$a_1 - 2v(v \cdot a_1) = ce_1$$

$$a_1 - ce_1 = 2v(v \cdot a_1)$$

$$v = \frac{a_1 - ce_1}{2(v \cdot a_1)}$$

I know that v is a unit vector...

$$\|v\| = 1$$

$$\|v\| = \left\| \frac{a_1 - ce_1}{2(v \cdot a_1)} \right\| = 1$$

Therefore $v = \dfrac{a_1 - ce_1}{\|a_1 - ce_1\|}$ ← now figure out what c is by plugging this in...

$$a_1 - 2vv^Ta_1 = ce_1$$

$$a_1 - 2\frac{a_1 - ce_1}{\|a_1 - ce_1\|}\left(\frac{a_1 - ce_1}{\|a_1 - ce_1\|}\right)^T a_1 = ce_1$$

$$a_1 - ce_1 - 2\frac{(a_1 - ce_1)(a_1 - ce_1)^T a_1}{\|a_1 - ce_1\|^2} = 0$$

$$\frac{(a_1 - ce_1)\|a - ce_1\|^2 - 2(a_1 - ce_1)(\|a_1\|^2 - ce_1 \cdot a_1)}{\|a_1 - ce_1\|^2} = 0$$

$$\|a_1 - ce_1\|^2 - 2(\|a_1\|^2 - ce_1 \cdot a_1) = 0$$

expand

$$\|a_1\|^2 - 2c\cancel{a_1 \cdot e_1} + c^2 - 2\|a_1\|^2 + 2c\cancel{e_1 \cdot a_1} = 0$$

Therefore $\|a_1\|^2 = c^2$ and $c = \pm\|a_1\|$

choose + or − so that the denominator $\|a_1 - ce_1\|$ is biggest.

Thus. $v = \dfrac{a_1 - ce_1}{\|a_1 - ce_1\|}$ $c = \pm\|a_1\|$