Newton's difference formula

$$f(x_j+\alpha h) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \Delta^k f_j$$ (lecture 21)

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Approximations

$$f(x_j+\alpha h) \approx \sum_{k=0}^{1} \binom{\alpha}{k} \Delta^k f_j = f_j + \binom{\alpha}{1}\Delta^1 f_j$$

$$= f_j + \alpha \Delta f_j$$

Now plug the interpolation in

$$\int_{x_j}^{x_{j+1}} f(x)\,dx = \int_0^1 \boxed{f(x_j+\vartheta h)}\, h\, d\vartheta$$

$$= h\int_0^1 (\underset{\uparrow\ \text{const}}{f_j} + \vartheta \underset{\uparrow\ \text{const}}{\Delta f_j})\, d\vartheta = h\left(f_j + \frac{\vartheta^2}{2}\Delta f_j\right)\Bigg|_0^1$$

$$= h\left(f_j + \frac{1}{2}\Delta f_j\right) = h\left(\cancel{f_j} + \frac{1}{2}\left(f_{j+1} - \cancel{f_j}\right)\right) = h\left(\frac{1}{2}f_j + \frac{1}{2}f_{j+1}\right)$$

$$= h\frac{f_j+f_{j+1}}{2}$$ ← trapezoid rule...

<u>Idea:</u> To approximate $\int_{x_j}^{x_{j+1}} f(x)\,dx$ first approximate f by a polynomial and then integrate that polynomial.

$n=2$

$$f(x_j+\alpha h)\approx\sum_{k=0}^{2}\binom{\alpha}{k}\Delta^k f_j = f_j+\alpha\Delta f_j+\binom{\alpha}{2}\Delta^2 f_j$$

$$= f_j+\alpha\Delta f_j+\frac{\alpha(\alpha-1)}{1\cdot 2}\Delta^2 f_j$$

$$\boxed{= f_j+\alpha\Delta f_j+\frac{\alpha^2-\alpha}{2}\Delta^2 f_j}$$

involves the $j+2$ term

$$\Delta f_j = f_{j+1}-f_j$$

$$\Delta^2 f_j=\Delta(f_{j+1}-f_j) = (f_{j+2}-f_{j+1})-(f_{j+1}-f_j) = f_{j+2}-2f_{j+1}+f_j$$

$$\frac{1}{h}\int_{x_j}^{x_{j+2}} f(x)\,dx = \frac{1}{\cancel{h}}\int_0^2 f(x_j+\vartheta h)\,\cancel{h}\,d\vartheta = \int_0^2\left(f_j+\vartheta\Delta f_j+\frac{\vartheta^2-\vartheta}{2}\Delta^2 f_j\right)d\vartheta$$

$x=x_j+\vartheta h$

$dx=h\,d\vartheta$

all these are constants...

$$= 2f_j+\frac{\vartheta^2}{2}\Delta f_j\Big|_0^2+\frac{\frac{\vartheta^3}{3}-\frac{\vartheta^2}{2}}{2}\Delta^2 f_j\Big|_0^2$$

$$= 2f_j+2\Delta f_j+\frac{8/3-\cancel{2}^{\,6/3}}{2}\Delta^2 f_j$$

$$= 2f_j + 2\Delta f_j + \frac{1}{3}\Delta^2 f_j$$

$$\approx 2\cancel{f_j} + 2\left(f_{j+1} - \cancel{f_j}\right) + \frac{1}{3}\left(f_{j+2} - 2f_{j+1} + f_j\right)$$

$$\approx \overset{6/3}{2} f_{j+1} + \frac{1}{3}\left(f_{j+2} - 2f_{j+1} + f_j\right)$$

$$= \frac{1}{3}\left(f_{j+2} + (6-2)f_{j+1} + f_j\right) = \frac{1}{3}\left(f_{j+2} + 4f_{j+1} + f_j\right) = S_j$$

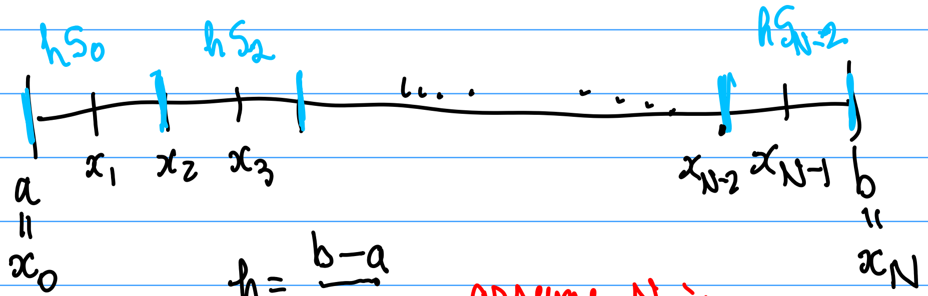
where $$S_j = \frac{1}{3}\left(f_j + 4f_{j+1} + f_{j+2}\right)$$

Therefore

2 points here

$$\int_{x_j}^{x_{j+2}} f(x)\,dx \approx h S_j$$

Now integrate over a larger interval



$$h = \frac{b-a}{N}$$

assume N is even so the ...'s work out here

Thus

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{N-2}}^{x_N} f(x)\,dx$$

$$= h\left(S_0 + S_2 + \dots + S_{N-2}\right)$$

$$= \frac{h}{3}\left(\left(f_0 + 4f_1 + f_2\right) + \left(f_2 + 4f_3 + f_4\right) + \dots + \left(f_{N-2} + 4f_{N-1} + f_N\right)\right)$$

$$= \frac{h}{3}\left(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \dots + 2f_{N-2} + 4f_{N-1} + f_N\right)$$

this is called Simpson's rule...

Error

$$\left|\int_a^b f(x)\,dx - \frac{h}{3}\left(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \dots + 2f_{N-2} + 4f_{N-1} + f_N\right)\right| \le \frac{(b-a)h^4}{180} \max_{a \le x \le b} \left|f^{(4)}(x)\right|$$

trapezoid method

$$\left|\int_a^b f(x)\,dx - \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \dots + 2f_{N-1} + f_N\right)\right| \le \frac{(b-a)h^2}{12} \max_{a \le x \le b} \left|f''(x)\right|$$

recall trapezoid method

$$\int_a^b f(x)\,dx \approx h\,\frac{f(x_0)+f(x_1)}{2} + h\,\frac{f(x_1)+f(x_2)}{2} + \dots + h\,\frac{f(x_{N-1})+f(x_N)}{2}$$

$$= \frac{h}{2}\left(\left(f_0 + f_1\right) + \left(f_1 + f_2\right) + \dots + \left(f_{N-1} + f_N\right)\right)$$

$$= \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \dots + 2f_{N-1} + f_N\right)$$

One could use the same idea to obtain even higher order approximations, but there is another improvement: the Gauss quadrature formula...

$$\int_{x_j}^{x_{j+1}} f(x)\,dx = \int_{-1}^{1} f\left(\tfrac{1}{2}(uh + x_j + x_{j+1})\right) \tfrac{1}{2}h\,du = \int_{-1}^{1} g(u)\,du$$

$$x = \tfrac{1}{2}(uh + x_j + x_{j+1}) \qquad dx = \tfrac{1}{2}h\,du$$

$$h = -1 \qquad x = \tfrac{1}{2}(-h + x_j + x_{j+1}) = x_j$$

$$h = 1 \qquad x = \tfrac{1}{2}(+h + x_j + x_{j+1}) = x_{j+1}$$

where $g(u) = f\left(\tfrac{1}{2}(uh + x_j + x_{j+1})\right)\tfrac{1}{2}h$

Now consider approximating

$$\int_{-1}^{1} g(u)\,du \approx w_1 g(x_1) + w_2 g(x_2)$$