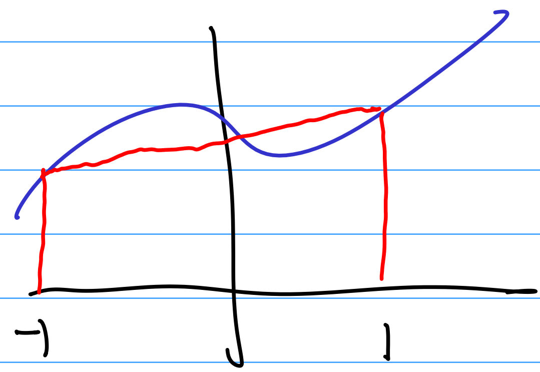
Now consider approximating

$$\int_{-1}^{1} g(u)\,du \approx \underbrace{w_1\, g(x_1) + w_2\, g(x_2)}_{\text{4 parameters here to optimize.}}$$

Trapezoid rule

$$\int_{-1}^{1} g(u)\,du = 2\,\frac{g(-1)+g(1)}{2} = g(-1)+g(1)$$



$$\left.\begin{aligned} w_1 &= 1 \\ w_2 &= 1 \\ x_1 &= -1 \\ x_2 &= 1 \end{aligned}\right\}\ \text{for trapezoid rule.}$$

$$\int_{-1}^{1} g(u)\,du \approx w_1\, g(x_1) + w_2\, g(x_2)$$

choose these in an optimal way to increase the order of the approximation.

Solve

this is all the trapezoid rule satisfies

$$\left\{\begin{aligned} \int_{-1}^{1} 1\,du &= \boxed{w_1 \cdot 1 + w_2 \cdot 1} \\ \int_{-1}^{1} u\,du &= \boxed{w_1 x_1 + w_2 x_2} \end{aligned}\right.$$

integrate constant function exactly

integrate linear function exactly

but there are two more parameters

$$\int_{-1}^{1} u^2\,du = w_1 x_1^2 + w_2 x_2^2$$ integrate quadratic functions exactly

$$\int_{-1}^{1} u^3\,du = w_1 x_1^3 + w_2 x_2^3$$ integrate cubic functions exactly

Suppose w_1, w_2, x_1, x_2 were chosen so all of the above hold.

$$p(x) = ax^3 + bx^2 + cx + d$$

$$\int_{-1}^{1} p(u)\,du = \int_{-1}^{1} (au^3 + bu^2 + cu + d)\,du$$

term by term

$$= a\int_{-1}^{1} u^3\,du + b\int_{-1}^{1} u^2\,du + c\int_{-1}^{1} u\,du + d\int_{-1}^{1} du$$

$$= a(w_1 x_1^3 + w_2 x_2^3) + b(w_1 x_1^2 + w_2 x_2^2) + c(w_1 x_1 + w_2 x_2) + d(w_1 \cdot 1 + w_2 \cdot 1)$$

$$= w_1(ax_1^3 + bx_1^2 + cx_1 + d) + w_2(ax_2^3 + bx_2^2 + cx_2 + d)$$

$$= w_1 p(x_1) + w_2 p(x_2)$$

Therefore

$$\int_{-1}^{1} g(u)\,du \approx w_1 g(x_1) + w_2 g(x_2)$$

is exact when g is any polynomial of degree 3 or less.

This is actually more accurate than simpson's rule.

$$\int_{-1}^{1} u^2\,du = \left.\frac{u^3}{3}\right|_{-1}^{1} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Solve for w_1, w_2, x_1, x_2 so that

First solve for w_1 and w_2 in terms of x_1 and x_2

$$2 = \int_{-1}^{1} 1\,du = \boxed{w_1 \cdot 1 + w_2 \cdot 1}$$

$$0 = \int_{-1}^{1} u\,du = w_1 x_1 + w_2 x_2$$

$$\frac{2}{3} = \int_{-1}^{1} u^2\,du = w_1 x_1^2 + w_2 x_2^2$$

$$0 = \int_{-1}^{1} u^3\,du = w_1 x_1^3 + w_2 x_2^3$$

$$x_1 2 = (\cancel{w_1 \cdot 1} + w_2 \cdot 1)x_1$$

$$0 = w_1 \cancel{x_1} + w_2 x_2$$

$$2x_1 = w_2(x_1 - x_2)$$

$$w_2 = \frac{2x_1}{x_1 - x_2} = \frac{-2x_1}{x_2 - x_1}$$

$$x_2 2 = (w_1 \cdot 1 + \cancel{w_2 \cdot 1})x_2$$

$$0 = w_1 x_1 + \cancel{w_2 x_2}$$

$$2x_2 = w_1(x_2 - x_1)$$

$$w_1 = \frac{2x_2}{x_2 - x_1}$$

Substitute into the last equation:

$$0 = w_1 x_1^3 + w_2 x_2^3 = \frac{\cancel{2}x_2}{\cancel{x_2 - x_1}} x_1^3 + \frac{-\cancel{2}x_1}{\cancel{x_2 - x_1}} x_2^3$$

Therefore $\cancel{x_2} x_1^{\cancel{3}\,2} = \cancel{x_1} x_2^{\cancel{3}\,2}$, $\quad x_1^2 = x_2^2 \quad$ so $\quad x_1 = \pm x_2$

but, since $x_1 \neq x_2$ then $x_1 = -x_2$

Plug into the last equation...

$$\frac{2}{3} = w_1 x_1^2 + w_2 x_2^2 = (w_1 + w_2)\, x_2^2$$

$$\frac{2}{3} = \left(\frac{2x_2}{x_2 - x_1} + \frac{-2x_1}{x_2 - x_1} \right) x_2^2$$

$$\frac{2}{3} = \Big(\underbrace{\frac{2x_2}{2x_2}}_{w_1} + \underbrace{\frac{2x_2}{2x_2}}_{w_2} \Big) x_2^2 = 2x_2^2$$

$$x_2^2 = \frac{1}{3}$$

Therefore

$$x_2 = \frac{1}{\sqrt{3}}, \quad x_1 = -\frac{1}{\sqrt{3}}, \quad w_1 = 1, \quad w_2 = 1$$

Consequently,

$$\int_{-1}^{1} g(u)\,du \approx g\left(\frac{-1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right)$$

Check.

```
julia> g0(x)=1; g1(x)=x; g2(x)=x^2; g3(x)=x^3
g3 (generic function with 1 method)
```

$$2 = \int_{-1}^{1} 1\,du = \boxed{w_1 \cdot 1 + w_2 \cdot 1}$$

$$0 = \int_{-1}^{1} u\,du = w_1 x_1 + w_2 x_2$$

$$\frac{2}{3} = \int_{-1}^{1} u^2\,du = w_1 x_1^2 + w_2 x_2^2$$

$$0 = \int_{-1}^{1} u^3\,du = \boxed{w_1 x_1^3 + w_2 x_2^3}$$

```
julia> g0(-1/sqrt(3))+g0(1/sqrt(3))
2

julia> g1(-1/sqrt(3))+g1(1/sqrt(3))
0.0

julia> g2(-1/sqrt(3))+g2(1/sqrt(3))
0.6666666666666669

julia> g3(-1/sqrt(3))+g3(1/sqrt(3))
0.0
```

Generalize this to to $n=3,4,5\ldots$ and so forth...

$$\int_{-1}^{1} g(u)\,du \approx \sum_{i=1}^{n} w_i\, g(x_i)$$

when $n=2$ the formula was exact up to cubics.

$n=3$ the formula was exact up to 5^{th} degree poly.

then n the formula was exact up to $2n-1$ degree poly.

We'll find x's and w's after the exam. Friday is Exam 2