Generalize this to to $n = 3, 4, 5 \ldots$ and so forth...

$$\int_{-1}^{1} g(u)\,du \approx \sum_{i=1}^{n} w_i\, g(x_i)$$

Orthogonal polynomials:

need a dot product between polynomials

Dot product between f and g

$$(f, g) = \int_{-1}^{1} f(x)\, g(x)\, dx$$

Norm of f

$$\|f\| = \sqrt{(f, f)} = \sqrt{\int_{-1}^{1} (f(x))^2\, dx}$$

Dot product and a norm is all that's need to run the Gram-Schmidt algorithm and that how you make things orthogonal...

Standard polynomial basis $\phi_0(x) = 1,\ \phi_1(x) = x,\ \phi_2(x) = x^2,\ \ldots,\ \phi_n(x) = x^n$.

Use Gram-Schmidt to make an orthonormal polynomial basis;

$p_0(x),\ p_1(x),\ \ldots,\ p_n(x)$..

$u_0 = \phi_0$ $\qquad$ $p_0 = \dfrac{u_0}{\|u_0\|}$

$u_1 = \phi_1 - (p_0, \phi_1)\, p_0$ $\qquad$ $p_1 = \dfrac{u_1}{\|u_1\|}$

$u_2 = \phi_2 - (p_0, \phi_2)\, p_0 - (p_1, \phi_2)\, p_1$ $\qquad$ $p_2 = \dfrac{u_2}{\|u_2\|}$

$\vdots$ $\qquad$ $\vdots$

$u_n = \phi_n - (p_0, \phi_n)\, p_0 - \cdots - (p_{n-1}, \phi_n)\, p_{n-1}$ $\qquad$ $p_n = \dfrac{u_n}{\|u_n\|}$

$$\begin{array}{c} \text{n+1 columns} \\ A = \left[\varphi_0 \middle| \varphi_1 \middle| \cdots \middle| \varphi_n \right] \end{array} \qquad \begin{array}{c} \text{n+1 columns} \\ \tilde{Q} = \left[p_0 \middle| p_1 \middle| \cdots \middle| p_n \right] \end{array}$$

$$\tilde{R} = \begin{bmatrix} \|u_0\| & (p_0, \varphi_1) & (p_0, \varphi_2) & & (p_0, \varphi_n) \\ & \|u_1\| & (p_1, \varphi_2) & & \vdots \\ & & \|u_2\| & & \\ & 0 & & \ddots & \\ & & & & \|u_n\| \end{bmatrix}$$

$$\tilde{R} \in \mathbb{R}^{(n+1)\times(n+1)}$$

The columns of $\tilde{Q}$ are orthonormal..

means if q is a polynomial of degree $n-1$ then

$$q = (p_0, q)p_0 + (p_1, q)p_1 + \cdots + (p_{n-1}, q)p_{n-1}$$

$$(q, p_n) = \big((p_0, q)p_0 + (p_1, q)p_1 + \cdots + (p_{n-1}, q)p_{n-1}, p_n\big) = 0$$

since $(p_i, p_n) = 0$ for $i = 0, 1, \ldots, n-1$.

To find the Quadrature formula..

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

Let x_i be the n roots of the orthogonal polynomial p_n and then choose the w_i so that the system of linear equation

$$\boxed{\int_{-1}^{1} \varphi_j(x)\,dx = \sum_{i=1}^{n} w_i \varphi_j(x_i) \quad \text{for } j = 0, \ldots, n-1}$$

n equations w/ n unknowns

Solved for the x's using orthogonal polynomials because those were appearing in a non-linear way. Then solve for the w's using the resulting system of linear equations.

Doesn't is work? Can we do it?

Theorem: $\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i)$ is exact for polynomials of degree $2n-1$.

Let p be a polynomial of degree $2n-1$ or less...

Need to check

exact equality...

$$\int_{-1}^{1} p(x)\,dx = \sum_{i=1}^{n} w_i p(x_i)$$

divide p by p_n ← orthogonal polynomial of degree n

$$p_n(x) \overline{\big)\, p(x)}^{\;q(x)}$$ remainder $r(x)$

Thus $\quad p(x) = p_n(x)\, q(x) + r(x)$.

$2n-1 \quad n \quad n-1 \quad n-1$

maximum degree of each term.

$$\int_{-1}^{1} p(x)\,dx = \int_{-1}^{1} \left(p_n(x)\, q(x) + r(x)\right) dx$$

$$= \underbrace{\cancel{\int_{-1}^{1} p_n(x)\, q(x)\, dx}}_{\text{dot product of } p_n \text{ with } q.} + \int_{-1}^{1} r(x)\, dx$$

Gram-Schmidt says $(p_n, q) = 0$

$$= \int_{-1}^{1} r(x)\,dx = \sum_{i=1}^{n} w_i\, r(x_i)$$

On the other hand

since x's are roots of P_n then $p_n(x_i) = 0$ for every i

$$\sum_{i=1}^{n} w_i\, p(x_i) = \sum_{i=1}^{n} w_i \left(\cancel{p_n(x_i)}\, q(x_i) + r(x_i)\right) = \sum_{i=1}^{n} w_i\, r(x_i)$$

Therefore

$$\int_{-1}^{1} p(x)\,dx = \sum_{i=1}^{n} w_i\, p(x_i)$$ for polynomials of degree less or equal $2n-1$.

Now actually do it ...

$\phi_0(x) = 1$, $\phi_1(x) = x$, $\phi_2(x) = x^2$, $\phi_3(x) = x^3$

and do Gram-schmidt.

$u_0 = \phi_0$ $\qquad p_0 = \frac{u_0}{\|u_0\|}$

$u_1 = \phi_1 - (p_0, \phi_1)p_0$ $\qquad p_1 = \frac{u_1}{\|u_1\|}$

$u_2 = \phi_2 - (p_0, \phi_2)p_0 - (p_1, \phi_2)p_1$ $\qquad p_2 = \frac{u_2}{\|u_2\|}$

$\vdots \qquad\qquad \vdots$

$u_n = \phi_n - (p_0, \phi_n)p_0 - \dots - (p_{n-1}, \phi_n)p_{n-1}$ $\qquad p_n = \frac{u_n}{\|u_n\|}$

$u_0 = 1$ $\qquad \|u_0\| = \sqrt{\int_{-1}^{1} (u_0(x))^2 dx} = \sqrt{\int_{-1}^{1} 1^2 dx} = \sqrt{2}$

Thus. $p_0(x) = \frac{1}{\sqrt{2}}$

$u_1 = \phi_1 - (p_0, \phi_1)p_0 = x - \left(\frac{1}{\sqrt{2}}, x\right)\frac{1}{\sqrt{2}} = x - \left(\int_{-1}^{1} \frac{1}{\sqrt{2}} x \, dx\right)\frac{1}{\sqrt{2}} = x$

$\|u_1\| = \sqrt{\int_{-1}^{1} x^2 dx} = \sqrt{\frac{x^3}{3}\Big|_{-1}^{1}} = \sqrt{\frac{2}{3}}$

Thus $p_1(x) = \frac{x}{\sqrt{2/3}}$

$$u_2 = \varphi_2 - (p_0, \varphi_2) p_0 - (p_1, \varphi_2) p_1$$

$$= x^2 - \left(\int_{-1}^{1} \frac{1}{\sqrt{2}} x^2 \, dx \right) \frac{1}{\sqrt{2}} - \left(\int_{-1}^{1} \frac{x}{\sqrt{2/3}} x^2 \, dx \right) \frac{x}{\sqrt{2/3}}$$

$$= x^2 - \frac{1}{\sqrt{2}} \frac{x^3}{3} \Big|_{-1}^{1} \cdot \frac{1}{\sqrt{2}} = x^2 - \frac{1}{3}$$

Note the solution of $x^2 - \frac{1}{3} = 0$ are the values of $x_1 = -\frac{1}{\sqrt{3}}$ and $x_2 = \frac{1}{\sqrt{3}}$ from the 2-point Gauss-quadrature formula from last week.