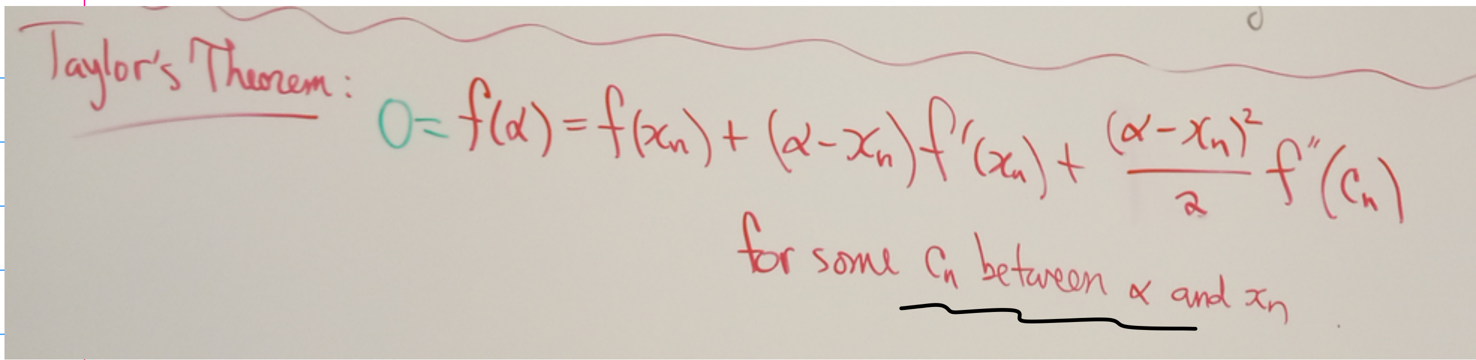


Taylor's Theorem. Suppose $f: \mathbb{R} \to \mathbb{R}$ with $n+1$ continuous derivatives, then

$$f(x) = \frac{(x-x_0)^0}{0!} f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \frac{(x-x_0)^3}{3!} f'''(x_0) + \cdots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + R_n(x)$$

where $R_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c)$ for some c between x and x_0

Usual proof uses the mean value theorem in a complicated way. Today's proof is with integration by parts.

Fundamental Theorem of Calculus

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(t)\,dt$$

$$f(x) = f(x_0) + R_0(x) \quad \text{where} \quad R_0(x) = \int_{x_0}^{x} f'(t)\,dt$$

rewrite this... so it looks more like Taylor's theorem

Assume $x > x_0$ (for definiteness)...

$$\underbrace{\min\{f'(t) : t \in [x_0, x]\}}_{m} \le f'(t) \le \underbrace{\max\{f'(t) : t \in [x_0, x]\}}_{M} \quad \text{for } t \in [x_0, x].$$

Thus $\quad m \le f'(t) \le M \quad$ for $\quad t \in [x_0, x]$.

$$\int_{x_0}^{x} m\,dt \le \int_{x_0}^{x} f'(t)\,dt \le \int_{x_0}^{x} M\,dt$$

$$(x-x_0)\,m \le \int_{x_0}^{x} f'(t)\,dt \le (x-x_0)\,M$$

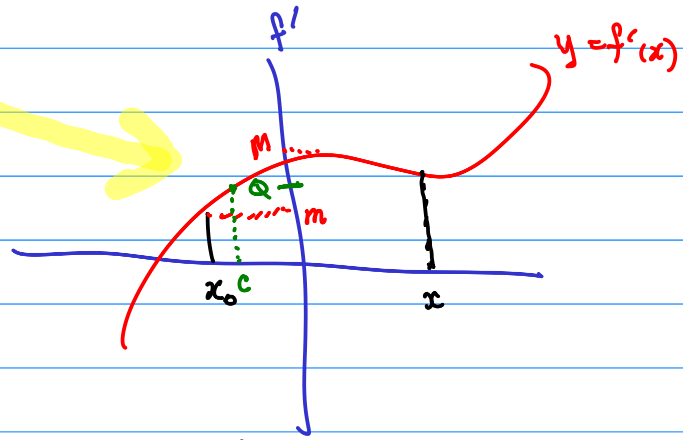
$$m \le \frac{\int_{x_0}^{x} f'(t)\,dt}{x-x_0} \le M$$

Q

this quotient is somewhere between the minimum and maximum of f'.
since f' is assumed continuous, then it takes all its values between the minimum and the maximum.

Therefore, there is a point $c \in [x_0, x]$ such that

$$f'(c) = \frac{\int_{x_0}^{x} f'(t)\,dt}{x-x_0}$$



Then

$$R_0(x) = \int_{x_0}^{x} f'(t)\,dt = (x-x_0) f'(c) \quad \text{for some } c \text{ between } x_0 \text{ and } x.$$

Taylor's theorem:

$$f(x) = f(x_0) + R_0(x) \quad \text{where } R_0(x) = (x-x_0) f'(c)$$

for some c between x_0 and x.

Now integrate the integral form of the remainder by parts

$$\int_{x_0}^{x} f'(t)\,dt = uv\Big|_{x_0}^{x} - \int_{x_0}^{x} v\,du = (t+t_0) f'(t)\Big|_{x_0}^{x} - \int_{x_0}^{x} (t+t_0) f''(t)\,dt$$

$u = f'(t)$ $du = f''(t)\,dt$

$dv = dt$ $v = t + t_0$ ← t_0 is a constant of integration.

Set $t_0 = -x$ then

Note: trying to get this $\frac{(x-x_0)f'(x_0)}{1!}$

$$(t+t_0)f'(t)\Big|_{x_0}^{x} = (x+t_0)f'(x) - (x_0+t_0)f'(x_0) = (x-x)f'(x) + (x-x_0)f'(x_0)$$

$-(x_0-x)$

$$\int_{x_0}^{x} f'(t)\,dt = (x_0-x)f'(x_0) - \int_{x_0}^{x} (t-x) f''(t)\,dt$$

or

$$\int_{x_0}^{x} f'(t)\,dt = (x_0-x)f'(x_0) + \int_{x_0}^{x} (x-t) f''(t)\,dt$$

plug in

Recall:

$$f(x) = f(x_0) + R_0(x) \quad \text{where} \quad R_0(x) = \int_{x_0}^{x} f'(t)\,dt$$

Thus

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + R_1(x) \quad \text{where} \quad R_1(x) = \int_{x_0}^{x} (x-t) f''(t)\,dt$$

Keep integrating by parts to get higher order approximations.

$$\int_{x_0}^{x} (x-t) f''(t)\,dt = -\frac{(x-t)^2}{2} f''(t)\Big|_{x_0}^{x} + \int_{x_0}^{x} \frac{(x-t)^2}{2} f'''(t)\,dt$$

$u = f''(t)$ $du = f'''(t)\,dt$

$dv = (x-t)\,dt$ $v = -\frac{(x-t)^2}{2}$ + const

$$\int_{x_0}^{x} (x-t) f''(t)\,dt = \frac{(x-x_0)^2}{2} f''(x_0) + \int_{x_0}^{x} \frac{(x-t)^2}{2} f'''(t)\,dt$$

2nd order Taylor

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2} f''(x_0) + R_2(x)$$

$$R_2(x) = \int_{x_0}^{x} \frac{(x-t)^2}{2} f'''(t)\,dt$$

↑ integral remainder.

by parts

$$\int_{x_0}^{x} \frac{(x-t)^2}{2} f'''(t)\,dt = -\frac{(x-t)^3}{3!} f'''(t)\Big|_{x_0}^{x} + \int_{x_0}^{x} \frac{(x-t)^3}{3!} f^{(4)}(t)\,dt$$

$u = f'''(t)$ $\quad du = f^{(4)}(t)\,dt$

$dv = \frac{(x-t)^2}{2}\,dt$ $\quad v = -\frac{(x-t)^3}{2 \cdot 3}$

$$\int_{x_0}^{x} \frac{(x-t)^2}{2} f'''(t)\,dt = \frac{(x-x_0)^3}{3!} f'''(x_0) + \int_{x_0}^{x} \frac{(x-t)^3}{3!} f^{(4)}(t)\,dt$$

3rd order

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2} f''(x_0) + \frac{(x-x_0)^3}{3!} f'''(x_0) + R_3(x)$$

where $R_3(x) = \int_{x_0}^{x} \frac{(x-t)^3}{3!} f^{(4)}(t)\,dt$

nth order

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2} f''(x_0) + \frac{(x-x_0)^3}{3!} f'''(x_0) + \cdots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0)$$

where $R_n(x) = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$

What's left is to write R_n in the familiar form for Taylor's theorem.