Thus

$$g(x_n, x_{n-1}) = x_n - \frac{f(x_n)(x_n - x_{n-1})}{\underbrace{f(x_n) - f(x_{n-1})}}$$

make sure this is not zero...

and Secant method is

$$x_{n+1} = g(x_n, x_{n-1})$$

Theorem 1.10 *Suppose that f is a real-valued function, defined and continuously differentiable on an interval $I = [\xi - h, \xi + h]$, $h > 0$, with centre point ξ. Suppose further that $f(\xi) = 0$, $f'(\xi) \neq 0$. Then, the sequence (x_k) defined by the secant method (1.25) converges at least linearly to ξ provided that x_0 and x_1 are sufficiently close to ξ.*

note $x_0 \neq x_1$

Proof: By the mean value theorem

$$f(x_n) - f(x_{n-1}) = f'(\varphi_n)(x_n - x_{n-1})$$

$$f(x_1) - f(x_0) = f'(\varphi_1)(x_1 - x_0)$$

for some φ_n between x_n and x_{n-1}.

φ_1 between x_1 and x_0

Let $\alpha = f'(\xi)$. Suppose for definiteness that $\alpha > 0$. If $\alpha < 0$ the argument is similar.

Since f' is assumed continuous there is a $\delta > 0$ such that

$$|f'(x) - f'(\xi)| < \frac{\alpha}{4} \quad \text{whenever} \quad |x - \xi| \leq \delta$$

$$|f'(x) - \alpha| < \frac{\alpha}{4}$$

by making δ smaller if necessary then can turn $<$ into $\leq$.

Thus

$$-\frac{\alpha}{4} < f'(x) - \alpha < \frac{\alpha}{4} \qquad \text{whenever } |x-\xi| \le \delta .$$

$$\frac{3\alpha}{4} < f'(x) < \frac{5\alpha}{4} \qquad \text{whenever } |x-\xi| \le \delta .$$

Therefore if x_0 and x_1 satisfy $x_0, x_1 \in I_\delta = [\xi-\delta, \xi+\delta]$.

then φ_1 between x_1 and x_0

means $\frac{3\alpha}{4} < f'(\varphi_1) < \frac{5\alpha}{4}$ and so $f'(\varphi_1) \neq 0$,

Then

$$f(x_1) - f(x_0) = \underbrace{f'(\varphi_1)(x_1 - x_0)}_{\text{since this is not zero}}$$

then $f(x_1) - f(x_0)$ is not zero so the secant method step

$$x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

makes sense.

Let $e_n = x_n - \xi$ be the error... Then

$$e_{n+1} = x_{n+1} - \xi = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} - \xi$$

$$= e_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

Since

$$f(x_n) - f(x_{n-1}) = f'(\varphi_n)(x_n - x_{n-1}) \qquad \text{substituting yields}$$

$$e_{n+1} = e_n - f(x_n)\frac{\cancel{x_n - x_{n-1}}}{f'(\varphi_n)(\cancel{x_n - x_{n-1}})}$$

$$e_{n+1} = e_n - \frac{f(x_n)}{f'(q_n)}$$

is similar...

Recall the Newton method analysis...

Estimate e_{n+1} in terms of e_n.

$$e_{n+1} = x_{n+1} - \alpha = x_n - \frac{f(x_n)}{f'(x_n)} - \alpha = e_n - \frac{f(x_n)}{f'(x_n)} = e_n - e_n + \frac{e_n^2}{2}\frac{f''(c_n)}{f'(x_n)}$$

For Newton we used Taylor's theorem 2nd order.

$$0 = \frac{f(x_n)}{f'(x_n)} - e_n\frac{f'(x_n)}{f'(x_n)} + \frac{e_n^2}{2}\frac{f''(c_n)}{f'(x_n)}$$

Easier 1st order Taylor...

$$0 = f(\xi) = f(x_n) + f'(c_n)(\xi - x_n)$$

use a higher order Taylor theorem for better estimates on the rate of convergence

Thus, $f(x_n) = -f'(c_n)(\xi - x_n) = e_n f'(c_n)$

for some c_n between x_n and ξ.

Again if x_0 and x_1 are close to ξ than c_1 will be equally close...

$$e_{n+1} = e_n - \frac{f(x_n)}{f'(q_n)} = e_n - \frac{e_n f'(c_n)}{f'(q_n)} = e_n\left(1 - \frac{f'(c_n)}{f'(q_n)}\right)$$

Note if q_1 and c_1 are close to ξ at the first step and thing are converging then they will be close to ξ at the next step... and so on...

Assume x_n and x_{n-1} are in $I_\delta = [\xi - \delta, \xi + \delta]$.

then $\frac{3\alpha}{4} < f'(q_n) < \frac{5\alpha}{4}$ and $\frac{3\alpha}{4} < f'(c_n) < \frac{5\alpha}{4}$

Thus

$$\frac{1}{\frac{3\alpha}{4}} > \frac{1}{f'(\varphi_n)} > \frac{1}{\frac{5\alpha}{4}} \quad \text{or} \quad \frac{4}{5\alpha} < \frac{1}{f'(\varphi_n)} < \frac{4}{3\alpha}$$

implies

$$\frac{3\alpha}{4}\frac{4}{5\alpha} < \frac{f'(c_n)}{f'(\varphi_n)} < \frac{5\alpha}{4}\frac{4}{3\alpha}$$

or

$$\frac{3}{5} < \frac{f'(c_n)}{f'(\varphi_n)} < \frac{5}{3}$$

$$1-\frac{5}{3} < 1-\frac{f'(c_n)}{f'(\varphi_n)} < 1-\frac{3}{5}$$

$$-\frac{2}{3} < 1-\frac{f'(c_n)}{f'(\varphi_n)} < \frac{2}{5}$$

Therefore

$$\left|1-\frac{f'(c_n)}{f'(\varphi_n)}\right| < \max\left\{\frac{2}{3},\frac{2}{5}\right\} = \frac{2}{3}$$

$$|e_{n+1}| = \left|e_n\left(1-\frac{f'(c_n)}{f'(\varphi_n)}\right)\right| \leq \frac{2}{3}|e_n|$$

So the error gets smaller at each iteration and the sequence $x_n \to \xi$ as $n \to \infty$.

The secant method actually converges faster than linearly...

With a bit more work one can actually show that

$$|e_{n+1}| \lesssim M|e_n||e_{n-1}|$$

for the secant method...

use a higher order Taylor theorem for better estimates on the rate of convergence

What can I infer about the order of the convergence from this?

I'm looking for the exponent β such that

$$|e_{n+1}| \approx C|e_n|^\beta$$

Then $$|e_n| \approx C|e_{n-1}|^\beta$$

$$|e_{n+1}| \approx C\left|C|e_{n-1}|^\beta\right|^\beta = C^{1+\beta}|e_{n-1}|^{\beta^2}$$

$$|e_{n+1}| \lesssim M|e_n||e_{n-1}|$$

$$C^{1+\beta}|e_{n-1}|^{\beta^2} \lesssim M\,C|e_{n-1}|^\beta\,|e_{n-1}|$$

If this holds true as $e_n \to 0$ then we need the same power of e_{n-1} on each side... Thus,

$$\beta^2 = \beta + 1 \quad \text{or} \quad \beta^2 - \beta - 1 = 0$$

$a=1,\ b=-1,\ c=-1$

$$\beta = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

After this heuristic argument we suspect

$$\beta = \frac{1+\sqrt{5}}{2}$$

might be the rate of convergence for Secant method. That is

$$|e_{n+1}| \lesssim C\,|e_n|^{\frac{1+\sqrt{5}}{2}}$$

```
julia> (1+sqrt(5))/2
1.618033988749895
```

Recall Newton's method was quadratic and so the number of correct digits about doubled at each iteration.

With the Secant method we expect 61% more correct digits at each iteration.