Secant Method,

this approximates $\frac{1}{f'(x_n)}$

$$x_{n+1} = x_n - f(x_n)\boxed{\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}}$$

Suppose $f(\xi)=0$ and x_0 and x_1 are close to ξ.

Let $e_n = x_n - \xi$ be the error.

$$e_n - e_{n-1} = x_n - \xi - (x_{n-1} - \xi) = x_n - x_{n-1}$$

$x_n = \xi + e_n$

$$e_{n+1} = x_{n+1} - \xi = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} - \xi$$

$$= \underbrace{x_n - \xi}_{e_n} - f(x_n)\frac{\overbrace{x_n - x_{n-1}}^{e_n - e_{n-1}}}{f(x_n) - f(x_{n-1})}$$

$$\approx e_n - \frac{f(x_n)(e_n - e_{n-1})}{f(x_n) - f(x_{n-1})} = \frac{e_n\left(f(x_n) - f(x_{n-1})\right) - f(x_n)(e_n - e_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$\approx \frac{e_{n-1} f(x_n) - e_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

want to factor out $e_n e_{n-1}$ and see the rest as bounded by a constant...

$$f(x_n) = f(\xi + e_n) = \underbrace{f(\xi)}_{0} + e_n f'(\xi) + \frac{e_n^2}{2} f''(\xi) + \frac{e_n^3}{3!} f'''(\xi) + \cdots$$

Known constants.

assume than e_n is small enough and that the series converges to $f(x_n)$

$$f(x_{n-1}) = f(\xi + e_{n-1}) = \underbrace{f(\xi)}_{0} + e_{n-1} f'(\xi) + \frac{e_{n-1}^2}{2} f''(\xi) + \frac{e_{n-1}^3}{3!} f'''(\xi) + \cdots$$

can this argument be adapted to avoid the infinite series and stop at maybe the second derivative with remainder.

$$f(x_n) - f(x_{n-1}) = e_n f'(\xi) + \frac{e_n^2}{2} f''(\xi) + \frac{e_n^3}{3!} f'''(\xi) + \cdots$$

$$- \left[e_{n-1} f'(\xi) + \frac{e_{n-1}}{2} f''(\xi) + \frac{e_{n-1}}{3!} f'''(\xi) + \cdots \right]$$

$$= (e_n - e_{n-1}) f'(\xi) + \left(\frac{e_n^2 - e_{n-1}^2}{2!} \right) f''(\xi) + \left(\frac{e_n^3 - e_{n-1}^3}{3!} \right) f'''(\xi) + \cdots$$

$$= (e_n - e_{n-1}) \left[f'(\xi) + \frac{e_n + e_{n-1}}{2!} f''(\xi) + \frac{e_n^2 + e_n e_{n-1} + e_{n-1}^2}{3!} f'''(\xi) + \cdots \right]$$

$$\begin{array}{r|l}
 & e_n^2 + e_n e_{n-1} + e_{n-1}^2 \\
\hline
e_n - e_{n-1} & e_n^3 - e_{n-1}^3 \\
 & e_n^3 - e_{n-1} e_n^2 \\
\hline
 & e_{n-1} e_n^2 - e_{n-1}^3 \\
 & e_{n-1} e_n^2 - e_n e_{n-1}^2 \\
\hline
 & e_n e_{n-1}^2 - e_{n-1}^3 \\
 & e_n e_{n-1}^2 - e_{n-1}^3 \\
\hline
 & 0
\end{array}$$

Thus

$$f(x_n) - f(x_{n-1}) = (e_n - e_{n-1}) \left[\underbrace{f'(\xi) + \overbrace{\frac{e_n + e_{n-1}}{2!} f''(\xi) + \frac{e_n^2 + e_n e_{n-1} + e_{n-1}^2}{3!} f'''(\xi) + \cdots}^{\to 0 \text{ as } e_n, e_{n-1} \to 0}}_{\substack{\text{bounded by a constant} \\ \text{and also away from zero} \\ \text{assuming the } e_n \text{ and } e_{n-1} \\ \text{are small enough..}}} \right]$$

$$e_{n-1} f(x_n) - e_n f(x_{n-1}) \approx e_{n-1}\left(e_n f'(\xi) + \frac{e_n^2}{2} f''(\xi) + \frac{e_n^3}{3!} f'''(\xi) + \cdots\right)$$
$$- e_n \left[e_{n-1} f'(\xi) + \frac{e_{n-1}^2}{2} f''(\xi) + \frac{e_{n-1}^3}{3!} f'''(\xi) + \cdots\right)$$

$$\approx \left(e_{n-1}\frac{e_n^2}{2} - e_n \frac{e_{n-1}^2}{2}\right) f''(\xi) + \left(e_{n-1}\frac{e_n^3}{3!} - e_n \frac{e_{n-1}^3}{3!}\right) f'''(\xi) + \cdots$$

$$= e_n e_{n-1}\left(\left[\frac{e_n}{2} - \frac{e_{n-1}}{2}\right] f''(\xi) + \left[\frac{e_n^2}{3!} - \frac{e_{n-1}^2}{3!}\right] f'''(\xi) + \cdots\right)$$

$$= e_n e_{n-1}(e_n - e_{n-1})\left(\underbrace{\tfrac{1}{2} f''(\xi)}_{\text{bounded constant}} + \underbrace{\frac{e_n + e_{n-1}}{3!} f'''(\xi) + \frac{e_n^2 + e_n e_{n-1} + e_{n-1}^2}{4!} f^{(4)}(\xi) + \cdots}_{\text{this part tends to zero as } e_n, e_{n-1} \to 0}\right)$$

$$e_{n+1} = \frac{e_{n-1} f(x_n) - e_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$= \frac{e_n e_{n-1} \cancel{(e_n - e_{n-1})}\left(\tfrac{1}{2} f''(\xi) + \frac{e_n + e_{n-1}}{3!} f'''(\xi) + \frac{e_n^2 + e_n e_{n-1} + e_{n-1}^2}{4!} f^{(4)}(\xi) + \cdots\right)}{\cancel{(e_n - e_{n-1})}\left[f'(\xi) + \frac{e_n + e_{n-1}}{2!} f''(\xi) + \frac{e_n^2 + e_n e_{n-1} + e_{n-1}^2}{3!} f'''(\xi) + \cdots\right]}$$

$$\approx e_n e_{n-1}\left(\frac{\tfrac{1}{2} f''(\xi) + \frac{e_n + e_{n-1}}{3!} f'''(\xi) + \frac{e_n^2 + e_n e_{n-1} + e_{n-1}^2}{4!} f^{(4)}(\xi) + \cdots}{f'(\xi) + \frac{e_n + e_{n-1}}{2!} f''(\xi) + \frac{e_n^2 + e_n e_{n-1} + e_{n-1}^2}{3!} f'''(\xi) + \cdots}\right)$$

Since $f'(\xi) \neq 0$ the stuff in the big parenthesis is bounded by a constant, therefore

$$|e_{n+1}| \leq M |e_n| |e_{n-1}| .$$

$\boxed{\checkmark}$

1.10 Write the secant iteration in the form

$$x_{k+1} = \frac{x_k f(x_{k-1}) - x_{k-1} f(x_k)}{f(x_{k-1}) - f(x_k)}, \qquad k = 1, 2, 3, \ldots .$$

Supposing that f has a continuous second derivative in a neighbourhood of the solution ξ of $f(x) = 0$, and that $f'(\xi) > 0$ and $f''(\xi) > 0$, define

Idea, to obtain that $|e_{k+1}| \leq M |e_k| |e_{k-1}|$ is we show $\varphi(x_k, x_{k-1})$ is bounded in a neighborhood of the root ξ.

$$\varphi(x_k, x_{k-1}) = \frac{x_{k+1} - \xi}{(x_k - \xi)(x_{k-1} - \xi)}, = \frac{e_{k+1}}{e_k e_{k-1}}$$

where x_{k+1} has been expressed in terms of x_k and x_{k-1}. Find an expression for

$$\psi(x_{k-1}) = \lim_{x_k \to \xi} \varphi(x_k, x_{k-1}),$$

and then determine $\lim_{x_{k-1} \to \xi} \psi(x_{k-1})$. Deduce that

$$\lim_{x_k, x_{k-1} \to \xi} \varphi(x_k, x_{k-1}) = f''(\xi)/2f'(\xi) .$$

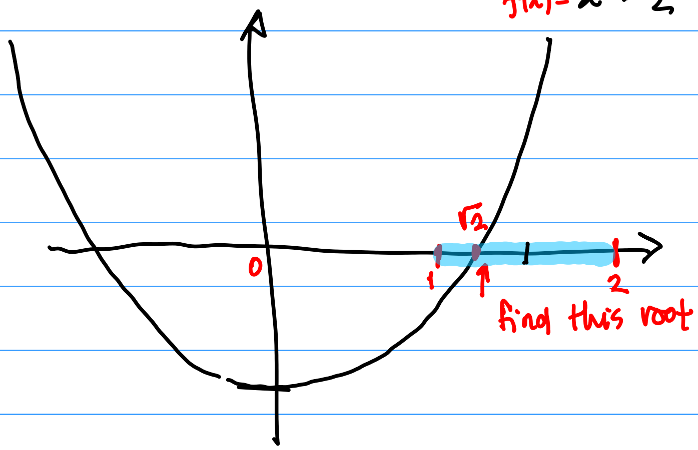
Same $\dfrac{\frac{1}{2} f''(\xi)}{f'(\xi)}$

There is a little bit of cheating here by assuming

$$\lim_{x_k, x_{k-1} \to \xi} \quad \text{is the same as} \quad \lim_{x_{k-1} \to \xi} \lim_{x_k \to \xi}$$

Bisection Method.

$f(x) = x^2 - 2$



$f(0) = -2$ too small

$f(2) = 4 - 2 = 2$ too big

Guess $c = 1$

$f(1) = 1 - 2 = -1$ too small

Guess $c = 1.5$

$f(1.5) = \frac{9}{4} - \frac{8}{4} = \frac{1}{4}$ too big

Next time use Julia...

$$\left.\begin{array}{l} \text{① Newton} \\ \text{② Secant} \\ \text{③ Relaxation} \\ \text{④ Bisection} \end{array}\right\} \text{Compare these}$$